KEY

TO THE

ELEMENTARY COURSE

OF

PRACTICAL MATHEMATICS,

CONTAINING

DEMONSTRATIONS OF THE RULES, AND FULL OR CONCISE SOLUTIONS OF MOST OF THE EXERCISES IN THE COURSE.

PART II.

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PRACTICAL GEOMETRY AND MENSURATION.

EDINBURGH: SUTHERLAND AND KNOX.

LONDON: SIMPKIN, MARSHALL & CO.

MDCCCLII.

MURRAY AND GIBB, PRINTERS EDINBURGH.

KEY

TO

PRACTICAL MATHEMATICS.

PART SECOND.

PRACTICAL GEOMETRY AND MENSURATION.

CONTENTS OF PART II.

PRACTICAL GEOMETRY AND MENSURATION.

CHAPTER II.

PRACTICAL GEOMETRY.

NOTE 1. Many of the propositions in this chapter agree so nearly with those in Euclid's Elements, or are so easily deduced from them, that the demonstrations will not be given in this volume, unless they differ materially from those of Euclid, or require some steps not very obvious. This remark, however, only applies to Practical Geometry: in Mensuration, the rules, though primarily depending on the elementary principles of Geometry, have no such close connection with them, and demand peculiar demonstrations of their own.

NOTE 2. In the course of the following pages, for the sake of conciseness, a few signs and abbreviations will be used in addition to those of Algebra, viz.—

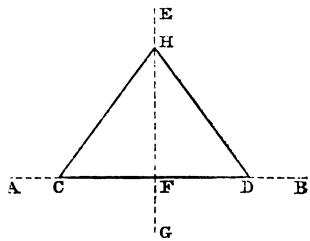
Straight.................................st.
Perpendicular...........................perp.
Angle ....$\angle$
Right angle ....$\llcorner$
Triangle.................................$\triangle$
Right-angled triangle..................$\triangle$

When we use the last symbol with letters, we understand that the middle letter is placed at the right angle. Thus $\triangle$ BCD, means the triangle BCD right-angled at C.

Proposition III.

Solution of the Exercise.

Draw any st. line AB, and on any part of it set off CD = 3, taken from any scale of equal parts. Bisect CD by a perp. EFG, and upon EF set off FH = 2, taken from the same scale. Join CH and HD. CHD will be the triangle required to be constructed.



HC and HD, being the two equal sides, will be found, by measurement on the same scale, to be each $2\frac{1}{2}$.

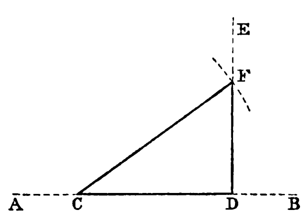
Proposition IV.

Demonstration of the Fifth Method.

O being the centre of the circle, DCE is a semicircle, and therefore the $\angle$ DCE is a right angle (Euc. III, 23), and CE is perp. to AB.

Solution of Exercise 2.

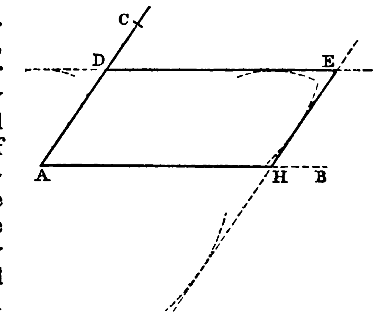
Draw any st. line AB, and on it set off CD = 4, taken from any scale of equal parts. Draw ED perp. to AB; and with the centre C and a radius = 5, taken from the same scale, cut the line ED in F. Join FC.



FC, measured on the same scale, will be found to be 3.

Proposition IX.

Exercise 2.

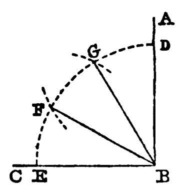


Draw any two st. lines, AB and AC, inclined to each other at any angle. Draw DE parallel to AB and at the distance of $\frac{1}{4}$ of an inch, by the directions given in the Problem. Also, by the same directions, draw EH parallel to AC and at the distance of 1 inch from it. ADEH will be the parallelogram required.

Proposition XIV.

Demonstration.

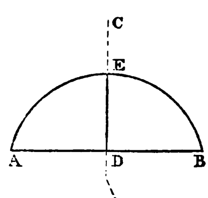
The three angles of an equilateral triangle are equal (Euc. I, 5, Cor.) The three angles of any triangle are together equal to two right angles. ∴ one angle of an equil. triangle is one-third part of two ∟s, or two-thirds of one ∟. Now if EG, in the diagram, were joined, EGB would be an equil. triangle, and consequently the ∠ GBE is two-thirds of a ∟. ∴ ABG is one-third of a ∟. In the same manner, by joining FD, we show that the ∠ EBF is one-third of a ∟. Consequently the ∠ FBG is the remaining third of the ∟ ABC.



Proposition XXIV.

Exercise 3.

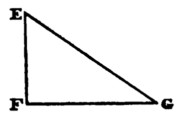
Draw any st. line and on it set off AB = 5. Bisect AB by the perp. CD, and on the latter set off DE = 2. Then, by the directions given, describe a circle passing through the points A, E, and B. Of that circle, AEB will be the arc required.



Proposition XXXVII.

Exercise 3.

Draw any st. line, and another perp. to it. On one of them set off EF = 2 inches. Then, since the $\angle$ G, opposite EF, is of 35°, the $\angle$ E must be of 55°. Therefore at the point E make an angle, FEG, of 55°, producing EG to cut FG in G. EFG will be the triangle directed to be made.



Proposition XXXVIII.

Exercise 1.

Since the three angles together contain 180 degrees, and the two angles at the base together contain 140 degrees, the angle at the vertex must contain 180 – 140, or 40, degrees.

Exercise 3.

Since the angle at the vertex is of 30°, the two angles at the base must together contain 150°, and, being equal, each of them must be an angle of 75°. Therefore, after setting off a base of 1 inch, make an angle of 75° at each extremity of it.

Proposition LI.

Exercise.

$360° \div 10 = 36° \doteq \angle$ AOB. (See diag. in Course.)

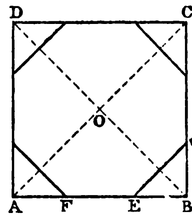
$$180° - 36° = 144° \doteq \angle OAB + \angle OBA.$$

$$\therefore \angle OAB \text{ or } OBA \doteq 72°.$$

Having, therefore, drawn a st. line AB = 1 inch, make the angles OAB and OBA each of 72°. With the centre O, and radius OA or OB, describe a circle, and, with the length AB in the compasses, step round the circumference, marking the various angular points of the decagon.

PROPOSITION LII.

Demonstration.



Let s be the side of any isosceles $\triangle$, and h the hypothenuse. Then (Euc. I, 47) $h^2 \doteq 2s^2$.

$\therefore\ h \doteq s \times \sqrt{2}$, and $s \doteq h \div \sqrt{2}$.

$\therefore$, in $\triangle$ EBG, $EG \doteq EB \times \sqrt{2}$.

and in $\triangle$ AOB, $AO \doteq AB \div \sqrt{2}$.

$\therefore\ AE = AO \doteq AB \div \sqrt{2}$.

$\therefore\ EB = AB - AE \doteq AB - AB \div \sqrt{2}$.

Multiplying the last equation first by 2 and then by $\sqrt{2}$, we have

$$2\,EB = 2\,AB - 2(AB \div \sqrt{2}) = 2\,AB - AB \times \sqrt{2},$$
$$\text{and } EG = EB \times \sqrt{2} = AB \times \sqrt{2} - AB.$$
$$\text{But } EF \doteq AB - 2\,EB = AB \times \sqrt{2} - AB.$$
$$\therefore\ EF \doteq EG.$$

And in the same way it may be proved that all the *sides* of the octagon are equal.

That all the *angles* are equal is evident; for the four triangles EBG &c., cut off from the four corners of the square, are all right-angled and isosceles, and therefore each of their acute angles is half a right angle, and consequently every one of the angles of the octagon is one right angle and a half.

PROPOSITION LVIII.

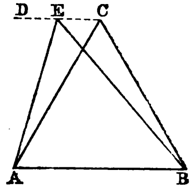
Exercise 2.

After constructing a triangle having its sides 5, 7, 10, take any one of these as the base, and draw a st. line through the vertex parallel to it. Bisect the base by a perp. meeting the parallel line in a point, say P, and draw st. lines from P to the extremities of the base.

Exercise 3.

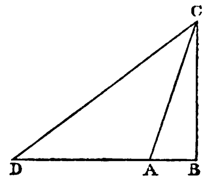
Divide the base into five equal parts, and from the four points of section draw st. lines to the vertex.

Exercise 4.



Having constructed the equil. △ ACB having each of its sides 1 inch, through C draw CD parallel to AB. Make the ∠ BAE of 75°, and join EB.

Exercise 5.



Having constructed the ⊾ ABC as directed, on AB produced set off AD = 3 times AB. Join DC. DCB will be the triangle required.

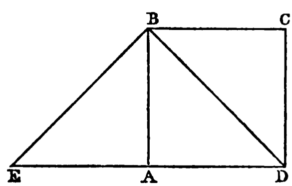
Proposition LIX.

Exercise 1.

Having constructed the triangle, draw a st. line through its vertex parallel to its base. Bisect the base, and from the point of section draw a st. line parallel to either side and meeting the parallel st. line previously drawn.

For the proof of the equality draw a st. line from the point of bisection to the vertex. That st. line divides the given triangle into two equal triangles, each of which is half of the given triangle, and, at the same time, half of the parallelogram constructed. ∴ the parallelogram is = the given triangle.

Exercise 2.



Having constructed a square ABCD on a base, AD, of 1 inch, produce AD to E, making AE = AD. Join EB and BD.

Exercise 3.

Having constructed the triangle described, through its *vertex draw* a st. line parallel to its base, and from one

extremity and the middle of the base draw two st. lines perp. to the base.

Proposition LXII.

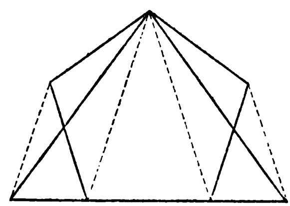
Demonstration.

The △ DLB ≐ △ DCB, both being on the same base, DB, and between the same parallels, DB and CL. Then since the latter of these two triangles is cut off from the given figure and the former added, the area of the figure remains the same.

Proposition LXIII.

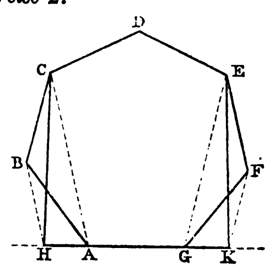
Exercise 1.

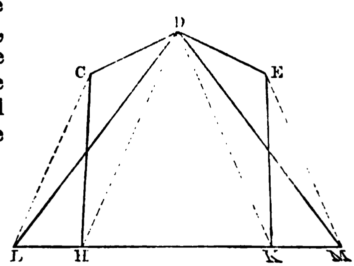
The work, complete, will be as in the annexed diagram; but, as mentioned in the note, the dotted lines need not be drawn.



Exercise 2.

ABCDEFG being the given heptagon, the first two steps in the process convert it into the figure HCDEK, and the next two, into the triangle LDM. The figures are here exhibited separate, to show distinctly the successive steps of the process, but in the actual operation both will be combined in one.





Exercise 3.

This will be done in the same manner as the preceding, by successive changes; but, in consequence of the re-entrant angle at H, the base will not *extend* at every step as in Exercise 2; but, at one step, the new extremity of the base will fall short of the extremity previously marked.

Proposition LXVI.

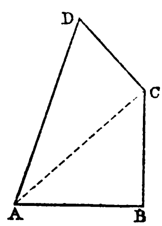
Exercise 1.

Answer, 50.

Exercise 2.

Construct a ⊾ having its base 30 and its hyp. 40, in the same manner as Ex. 2 of Prop. IV. Then describe a square on its hypothenuse.

Exercise 3.



After constructing a ⊾ ABC, having its base AB = 16, and its perp. BC = 15, draw CD perp. to AC, and = 12. Join AD, and upon it describe a square.

We shall find that AD ≐ 25.

Exercises 4 *and* 5.

These are intended to be done by following out the same process, but, of course, there is another way by which both may be done more easily.

Corollary 2.

Exercise 1.

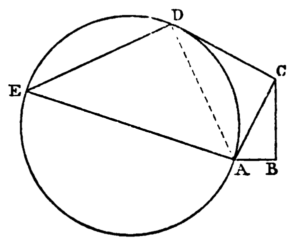
Construct a ⊾ having its two legs 12 and 16, and on the hypothenuse describe an equilateral triangle. The side *will be 20.*

Exercise 2.

Having constructed the figure ABCD as in Exercise 3, above, making AB, BC, and CD each = $\frac{1}{2}$ inch, describe a hexagon on AD.

Exercise 3.

Constructing our diagram in the same manner as in all the preceding exercises, making AB = 1, BC = 2, CD = 3, and DE = 4, describe a circle on AE as a diameter.



PROPOSITION LXVII.

Exercises 1 *and* 2 are performed exactly according to the general direction given for the construction of the problem, only that, in the latter, AB must be the given diameter, the diameter of the required circle being AD.

Exercise 3 is intended to be done by first going through the same process, and then repeating it on the line AD. We ultimately obtain a line = AC, or half of AB, a result which, of course, may be obtained much more easily by a subsequent proposition.

PROPOSITION LXIX.

Exercise 1.

In this instance DG being set off = 9 and DH = 12, the point H will be beyond G, that is, further from D, and consequently the point K beyond I.

The same remark applies to *Exercises* 2 *and* 3.

Exercise 3.

In this instance we have DG = 20, DH = 30, and DI = 30. The third proportional required, viz. DK, will be 45.

Proposition LXXI.

Exercise 1.

In this exercise we have AB = 4, and BE = 9. The result is BF = 6.

Arithmetically the same result is found by taking the square root of 4×9, that is, of 36.

Exercise 2.

The mean proportional is found to be 162.

Propositions LXXXII, LXXXIII, and LXXXIV.

The demonstration of the first of these is given under Problem XXVI of Capter III; that of the second, under Pr. L of Ch. IV; and that of the remaining one, under Pr. XLVII of Ch. V.

Proposition LXXXVIII.

Demonstration.

The point F is taken so as to divide the curvature of the elliptic quadrant into two equal portions, the tangent at that point making an angle of 45° with each axis. This is done in order that one half of the curvature may be described with the one centre, and the other half with the other: but although the point F is correctly ascertained for that purpose in the true ellipse, yet the object is only approximately attained in the figure actually drawn.

Call the semiaxes a and b and their parallel co-ordinates in the true ellipse, x and y. Then, by that property of the ellipse which is usually employed to express its algebraic equation,

$$a^2y^2 \doteq b^2(a^2 - x^2)$$

Hence, by the Differential Calculus,

$$a^2y \,\partial\, y \doteq -b^2x \,\partial\, x.*$$

* This peculiar form of the letter *d* is used, throughout this volume, to express the differential, in order to avoid all ambiguity when *d* happens to be used in its common form for other purposes, and to show at once that a differential is intended.

But since the point F is required to be such that the tangent at that point shall be equally inclined to both axes, we have consequently, at that point,

$$\partial y = -\partial x.$$

$$\therefore a^2 y \partial x \doteq b^2 x \partial x, \text{ or } a^2 y = b^2 x.$$

$$\text{Hence } a^4 y^2 = a^2 \times a^2 y^2 \doteq b^4 x^2.$$

From this and the first equation,

$$a^2 \times b^2(a^2 - x^2) \doteq b^4 x^2; \text{ or } a^4 - a^2 x^2 \doteq b^2 x^2.$$

$$\therefore a^4 \doteq (a^2 + b^2) x^2; \text{ or } a^2 \doteq \sqrt{(a^2+b^2)} \times x.$$

$$\text{Hence } x \doteq \frac{a^2}{\sqrt{(a^2+b^2)}}; \text{ and } \sqrt{(a^2+b^2)} : a :: a : x.$$

But, since $a^2 y \doteq b^2 x$,

$$y \doteq \frac{b^2}{a^2} x = \frac{b^2}{a^2} \times \frac{a^2}{\sqrt{(a^2+b^2)}} = \frac{b^2}{\sqrt{(a^2+b^2)}}.$$

$$\therefore \sqrt{(a^2+b^2)} : b :: b : y.$$

Now the values of x and y, expressed by the two proportions just ascertained, will be found realized in the geometrical construction, for (Euc. VI, 8, Cor.)

BA : AC :: AC : AD (= CE by construction).

(See the diagram in the Course.)

That is, $\sqrt{(a^2+b^2)} : a :: a :$ CE or x.

And BA : BC :: BC : BD (= EF by construction).

That is, $\sqrt{(a^2+b^2)} : b :: b :$ EF or y.

Lastly, the arc first described will pass through F, because AO is evidently equal to OF (AOF being an isosceles triangle): and the arc BH described with the radius Bo, will meet the other arc in H, because oG ≐ oO, and GB = AO ≐ OH.

Proposition XC.

The mechanical construction by the method described, depends on a well-known property of the ellipse, that the sum of any two lines drawn from any point in the circumference to the foci is a constant quantity. When this is made the definition of the ellipse, no *demonstration*, of course, is required. But when the ellipse is defined by any of its other properties, the demonstration of this, as a theorem, depends on the definition given. Such a demonstration may be found in every treatise on the Conic Sections in which that property is not employed as the fundamental one for the ellipse. The definition of an ellipse given in this work derived from the section of a cone

is rather intended as an explanation than as a definition, and is not the best for founding demonstration upon.

Problem CII.

Demonstration.

The point to be demonstrated is, that if the arc EGF be described according to the direction given, it will be equal to the circumference of the circle forming the base of the cone, that is, of a circle whose diameter is AB.

Allowing C to represent the circumference of the whole circle of which EGF forms a part, and c, that of the circle forming the base of the cone. Since the circumferences of two circles are to each other as their radii,

$$C : c :: \text{CB} : \text{BD},$$
$$\text{and } \tfrac{1}{4}\, C : \tfrac{1}{2} c :: \text{CB} : \text{AB}.$$
$$\text{But } C : \text{arc EGF} :: 360° : \angle\,\text{ECF (Euc. vi, 33)}.$$
$$\therefore\ \tfrac{1}{4}\, C : \tfrac{1}{2} \text{ arc EGF} :: 90° : \angle\,\text{ECG}.$$
$$\text{But, by construction, CB} : \text{AB} :: 90° : \angle\,\text{ECG}.$$
$$\therefore\ \tfrac{1}{4}\, C : \tfrac{1}{2} \text{ arc EGF} :: \text{CB} : \text{AB}.$$
$$\therefore\ \tfrac{1}{4}\, C : \tfrac{1}{2} c :: \tfrac{1}{4}\, C : \tfrac{1}{2} \text{ arc EGF}.$$
$$\therefore\ \text{Arc EGF} \doteq c = \text{circumf. of cone's base.}$$

CHAPTER III.

MENSURATION OF LINES.

Problem I.

The *demonstration of the rule* follows directly from Euclid I, 47.

Exercise 1.

$$\begin{array}{r} 513^2 \doteq 263{,}169 \\ 684^2 \doteq 467{,}856 \\ \hline \text{Hyp.}^2 \doteq 731{,}025. \end{array}$$

Exercise 2.

Reducing both dimensions to inches, we have Perp.$^2 = 34^2 - 30^2 = 1156 - 900 \doteq 256$.

Exercise 3.

$$\begin{array}{rr} \text{Side}^2 \doteq & 95{\cdot}531076 \\ & 2 \\ \hline \text{Diagonal}^2 \doteq & 191{\cdot}062152. \end{array}$$

Exercise 4.

$$\text{Diagonal}^2 = 101^2 \doteq 10201{\cdot}$$
$$\text{Side}^2 \doteq 10201 \div 2 = 5100{\cdot}5.$$

Exercise 5.

$$\begin{array}{lr} \text{Perp.}^2 = 67{\cdot}5^2 & \doteq 4556{\cdot}25. \\ \text{Base}^2 = 36^2 & \doteq 1296{\cdot}00 \\ \hline \text{Hyp.}^2 & \doteq 5852{\cdot}25. \end{array}$$

Exercise 6.

When we draw the diagonal, the field is, of course, divided into two ⊿s, in either of which the hypothenuse is the diagonal of the field, the base being one of the sides of the field, and the perpendicular the other.

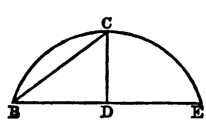
$$\begin{array}{lr} \text{Hyp.}^2 = 10{\cdot}71^2 & \doteq 114{\cdot}7041 \\ \text{Base}^2 = 9{\cdot}45^2 & \doteq 89{\cdot}3025 \\ \hline \text{Perpendicular}^2 & \doteq 25{\cdot}4016. \end{array}$$

Exercise 7.

A perp. from the vertex to the base divides the isosceles △ into two ⊿s, in either of which we have the hyp. = 65 and the base = 25. From these data we find the perp. = 60.

Exercise 8.

In the ⊿ BDC we have given BD = 5, CD = 4. Hence, by the rule, we find BC = $\sqrt{41}$.

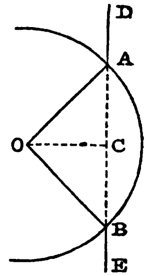


Exercise 9.

In this question we have given BC = 11·3 and BD = 8·25. CD is required.

$$\begin{array}{lr} BC^2 \doteq & 127{\cdot}6900 \\ BD^2 \doteq & 68{\cdot}0625 \\ \hline CD^2 \doteq & 59{\cdot}6275. \end{array}$$

Exercise 10.



Let O be the centre of the axle; OA or OB, the shaft; and DE, the wall: then AB is the extent of wall which must be taken down.

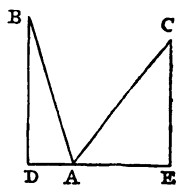
Now in the $\triangle$ ACO, AO $\doteq$ 15, and OC $\doteq 10\frac{1}{2}$. $\therefore$, by the rule, AC $\doteq$ 10·71 +, and AB = twice AC $\doteq$ 21·4 +.

Exercise 11.

Let AB and AC be the ladder in its two positions, and DE the street. Then,

In $\triangle$ ADB,	In $\triangle$ AEC
AB $\doteq$ 50,	AC $\doteq$ 50,
BD $\doteq$ 48.	CE $\doteq$ 40.
$\therefore$ AD $\doteq$ 14.	$\therefore$ AE $\doteq$ 30.

$\therefore$ DE = 14 + 30 $\doteq$ 44.



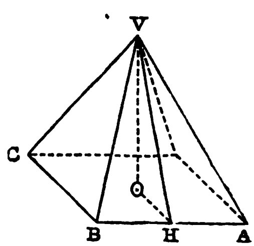
Exercise 12.

$$\begin{aligned} \text{Here } h &\doteq 100, \text{ and } p \doteq 99{\cdot}5 \\ \therefore h^2 &\doteq 10000 \\ \text{and } p^2 &\doteq \underline{9900{\cdot}25} \\ \therefore b^2 &\doteq 99{\cdot}75. \end{aligned}$$

Exercise 13.

Let O be the centre of the base. The Δ VOH is right-angled at O. (See the model.) In that triangle, VO $\doteq$ 100, and OH = 50. $\therefore$ VH^2 $\doteq$ 12500 and VH $\doteq$ 111·8 +.

Again the Δ VHB being right-angled at H, and VH^2 = 12500, and HB^2 = 50^2 = 2500, *VB^2 $\doteq$ 15000,* and VB = 122·5.



Exercise 14.

If a perp. be drawn from the vertex to the base, it divides the equil. Δ into two $\triangle$s, in each of which, $h \doteq 1$ and $b \doteq \cdot 5$, giving $p^2 = \cdot 75$.

PROBLEM VI.

Demonstration of the Rule.

The number Π is the circumference of a circle whose diameter is 1, from which that of any other circle may be found. For, since the circumferences of any two circles are to each other as their diameters (Playfair's Geometry, Supplement, B. I, Pr. VI), $1 : D :: \Pi : C$

$$\therefore C \doteq D \times \Pi.$$

NOTE. Archimedes, the famous Mathematician of Syracuse, was the first to compute the ratio of the diameter of a circle to the circumference with any considerable degree of accuracy. He proves that it must be the ratio of 1 to a number somewhere between $3\frac{10}{70}$ and $3\frac{10}{71}$*. The former of these, or the ratio of 7 to 22, forms a convenient practical rule, and when expressed decimally is correct to *three* figures, or two decimal places. In the fifth century of our era, Sporus extended the calculation as far as 10,000th parts, which is equivalent, decimally, to *four or five* figures. The introduction of the Arabic or modern notation into Arithmetic, much facilitated this, as well as all other calculations. In the end of the sixteenth century the number Π was computed by Vieta, a French Mathematician, as far as *eleven* figures; and, soon after this, it was simplified in its form of expression, and still farther extended, by Metius, Adrianus Romanus, and Van Ceulen, all natives of the low countries, —the first reducing it to the very simple fractional value, $\frac{355}{113}$, which, expressed decimally, is correct to *seven* figures, —the second, Adrianus, carrying the decimal expression to *seventeen* figures, and Van Ceulen to *thirty-six*, namely, 3·141,592; 653,589; 793,238; 462,643; 383,279; 502,88 +. The last was regarded as so wonderful an attainment that the number was engraved on his tombstone. This achievement of unwearied industry could not then be surpassed,

* The method employed for this purpose by Archimedes and those who followed him, till the discovery of Fluxions, will be shown in this volume, Chapter IV, Pr. XIII.

and the number, though verified by Snellius (also a Dutchman), was never extended farther till Sir Isaac Newton's discovery of Fluxions opened the way to methods so much more expeditious, that Van Ceulen's result, which cost two years of constant toil, can now be attained in a few hours. It was subsequently extended by Machin, in England, to *one hundred* figures, by De Lagney and others, in France, still farther, and recently, by Dr Rutherford of Woolwich, and Mr Shanks of Houghton-le-Spring, to 421 figures.

Methods of calculating the Number Π.

I. We know that the circumference is less than that of any polygon circumscribed about it, and greater than that of any polygon inscribed in it. We may, therefore, describe a square in a circle whose diameter is 1, and another square about the same circle, the circumferences of which squares will be 2·8 + and 4·0. The circumference of the circle must be somewhere between these; but, as they differ so far, they are of no use. We therefore find, next, the circumferences of two octagons described in the same manner, which are 3·06 + and 3·31 +. As these agree only in the first figure, we have, as yet, found no more than one figure correct. We therefore proceed in the same way with polygons of 16, 32, 64, &c. sides. Or, instead of the square, we may commence with the hexagon, and proceed from it to polygons of 12, 24, 48, &c. sides.* This method, however (whether we start with the square or the hexagon), is very tedious. We therefore go on to the others.

II. We find the area of the circle in the manner explained in this volume, Ch. IV, Pr. XIII, and divide the area thus found by half the radius. This method is also very laborious.

III. By the method of *Fluxions*, or, in other words, by the Differential and Integral Calculus, thus:—

Let z be any arc of a circle and x its tangent, the radius being 1. Then, as may be found in any treatise on the Differential Calculus,

$$\partial z \doteq \frac{\partial x}{1+x^2} = \partial x \times \frac{1}{1+x^2}.$$

$$\text{But } \frac{1}{1+x^2} \doteq 1 - x^2 + x^4 - x^6 + \&c.$$

$$\therefore \partial z \doteq \partial x - x^2\,\partial x + x^4\,\partial x - x^6\,\partial x + \&c.$$

* A very neat mode of doing this will be found in Hutton's *Course*, Mensuration, Pr. VII.

By integrating the successive terms of this series, we obtain (the constant being = 0).

$$z = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c.$$

Having found this *general* formula, we must, in order to apply it to our purpose, give a *particular* value to x and z. Let us take for z an arc of 45°, or $\frac{1}{8}$ of the circumference. Then $x = \tan 45° \doteq 1$.

$$\therefore \text{Arc of } 45° \doteq 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \&c.$$

Having calculated the sum of this series to any extent required, we multiply that sum by 4, for the semicircumf. to the radius 1, which is equal to the whole circumf. to the diameter 1, or the number Π. The series is very simple and very easily calculated; but it converges so slowly, that, to obtain a result correct only to a few figures, the number of terms required is immense, and the labour very great.

To obtain a more rapidly converging series, let us take an arc of 30°, the tangent of which is $1 \div \sqrt{3}$, and let us, at the same time, make a slight change in the mode of expressing the general formula, thus—

$$z = x\,(1 - \tfrac{1}{3}x^2 + \tfrac{1}{5}x^4 - \tfrac{1}{7}x^6 + \&c.)$$

When the arc is of 30°, x being then the tangent of 30°, is easily found to be $1 \div \sqrt{3}$; for $\sin 30° \doteq \frac{1}{2}$, $\cos 30° = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$, and cos : sin :: rad : tan, or $\frac{1}{2}\sqrt{3} : \frac{1}{2} :: 1 : 1 \div \sqrt{3}$.

$$\therefore x \doteq \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3},\ x^2 = \frac{1}{3},\ x^4 = \frac{1}{3^2},\ x^6 = \frac{1}{3^3},\ \&c.$$

$$\therefore \text{Arc of } 30° \doteq \frac{1}{3}\sqrt{3}\,(1 - \frac{1}{3 \cdot 3^1} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \&c.),$$

and semicircumf. to rad. 1 = circumf. to diam. 1 =

$$\Pi \doteq 2\sqrt{3}\,(1 - \frac{1}{3 \cdot 3^1} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \&c.)$$

The last formula, first proposed by Halley, was used by Abraham Sharp in calculating the circumference to 72 places of figures, by Machin in calculating it to 100, and by De Lagney, to 128.

The actual process of computation is as follows:—

Putting Q to represent the quotient of $1 \div 3$, or the fraction $\frac{1}{3}$, Halley's formula becomes

$$\Pi = 2\sqrt{3} \times (1 - \tfrac{1}{3}\,Q + \tfrac{1}{5}\,Q^2 - \tfrac{1}{7}\,Q^3 + \tfrac{1}{9}\,Q^4 - \&c.)$$

We therefore, first of all, find the values of Q, Q^2, Q^3, &c. each of which we carry to seven decimal places, that we may have the result true to six figures—the extent to which we now propose to carry our calculation.

3)1·000,000,0
3) ·333 333 3 $= Q$
3) ·111 111 1 $= Q^2$
3) ·037 037 0 $= Q^3$
3) ·012 345 7 $= Q^4$
3) ·004 115 2 $= Q^5$
3) ·001 371 7 $= Q^6$
3) ·000 457 2 $= Q^7$
3) ·000 152 4 $= Q^8$
3) ·000 050 8 $= Q^9$
3) ·000 016 9 $= Q^{10}$
3) ·000 005 6 $= Q^{11}$
3) ·000 001 9 $= Q^{12}$
·000 000 6 $= Q^{13}$

3)·333,333,3
·111 111 1 $= \frac{1}{3} Q$.
5)·111 111 1
*22 222 2 $= \frac{1}{5} Q^2$.
7) 37 037 0
5 291 0 $= \frac{1}{7} Q^3$.
9) 12 345 7
1 371 7 $= \frac{1}{9} Q^4$.
11) 4 115 2
374 1 $= \frac{1}{11} Q^5$.
13) 1 371 7
105 5 $= \frac{1}{13} Q^6$.
15) 457 2
30 5 $= \frac{1}{15} Q^7$.

17)152,4
9 0 $= \frac{1}{17} Q^8$

19)50,8
2 7 $= \frac{1}{19} Q^9$

21)16,9
8 $= \frac{1}{21} Q^{10}$

23)5,6
2 $= \frac{1}{23} Q^{11}$

25)1,9
1 $= \frac{1}{25} Q^{12}$

We now proceed to collect the terms already computed.

Positive Terms.		*Negative Terms.*	
$1 =$	1·000,000,0	$\frac{1}{3} Q =$	·111,111,1
$\frac{1}{5} Q^2 =$	22 222 2	$\frac{1}{7} Q^3 =$	5 291 0
$\frac{1}{9} Q^4 =$	1 371 7	$\frac{1}{11} Q^5 =$	374 1
$\frac{1}{13} Q^6 =$	105 5	$\frac{1}{15} Q^7 =$	30 5
$\frac{1}{17} Q^8 =$	9 0	$\frac{1}{19} Q^9 =$	2 7
$\frac{1}{21} Q^{10} =$	8	$\frac{1}{23} Q^{11} =$	2
$\frac{1}{25} Q^{12} =$	1	$\frac{1}{27} Q^{13} =$	0
Sum $=$	$+$1·023 709 3	Sum $=$	·116 809 6.
	$-$0·116 809 6		
	·906 899 7		

* It is not thought necessary any farther to insert the point and the cyphers, as all the lines terminate together.

$$
\begin{array}{l}
\begin{array}{r}
\cdot 906{,}899{,}7 \\
8\ 050\ 237\cdot 1
\end{array} = \sqrt{3}\text{, reversed.} \\
\hline
\left.\begin{array}{r}
906\ 899\ 7 \\
634\ 829\ 8 \\
27\ 207\ 0 \\
1\ 813\ 8 \\
45\ 3 \\
7
\end{array}\right\}\text{ By contracted multiplication.} \\
\hline
\begin{array}{r}
1\cdot 570\ 796\ 3 \\
2 \\
\hline
3\cdot 141\ 592\ 6
\end{array} = \Pi.
\end{array}
$$

The result we have obtained is carried to eight figures; but as the sum of the series was carried only to seven figures, we cannot regard more than six as certain, the seventh being only probably correct, and the eighth, though right, a mere chance figure.

To carry the result twice as far, would require about four times the labour; to carry it three times as far, would require about nine times as much; and to obtain the result correct to 36 figures (Van Ceulen's extent), would call for an amount of figures and of labour about 36 times as great as the work exhibited above: 72 figures (Sharp's extent), would involve 144 times as much, and 128 figures (De Lagney's limit) could not be done with less than 450 times.

After the great extent to which the expression was carried by the calculators already mentioned, any simplification of the mode could only be a subject of curiosity, as there is no probability that the result can ever be required to any extent even approaching that which they attained. After Machin had finished his task, however, he showed how it might have been effected much more easily by a peculiar mode of applying the general formula, and other authors have shown similar modes. Thus Euler pointed out that the arc whose tangent is $\frac{1}{2}$, and the arc whose tangent is $\frac{1}{3}$, together make an arc of 45°. If therefore the length of these two arcs be computed and added together, and the sum be multiplied by 4, we shall then have the number Π, and the two series will both converge more quickly than that for the arc of 30°. Machin proposed to compute the arc whose tangent is $\frac{1}{5}$, and then the arc whose tangent is $\frac{1}{239}$, and showed that the latter of these, taken from four times the former, would make an arc of 45°.

Dr Hutton, in his *Tracts*, has shown a general method of which Euler's and Machin's are only particular cases. That method will be found fully detailed in Tract 18 of Vol. 1, and also in the Key to the Author's "Complete Treatise," page 22 to 25, in which the best of Hutton's particular method is considerably improved for use and exhibited in actual calculation.

The process which Dr Rutherford has employed is still more concise than any of them,* when the calculation is required to be carried to any great extent, and when the easy mode of dividing by 99 is employed. Dr Rutherford proceeds upon a formula of Euler's, which is, that an arc of $45° \doteq 4$ times an arc whose tangent is $\frac{1}{5}$ − an arc whose tangent is $\frac{1}{70}$ + an arc whose tangent is $\frac{1}{99}$. For more full details of Dr Rutherford's method, see the Philosophical Transactions for 1841, and the new Course of Mathematics for the use of the Royal Military Academy.

Exercise 5.

Radius $\doteq 29\frac{1}{2}$. ∴ Diam. $\doteq 59$ inches.
Hence we find circumf. $= 185{\cdot}3544$.

Exercise 6.

For this we use the reverse formula, $D = C \div \pi$. That is, we divide 6·224 by 3·1416. If we use the second rev. formula, we multiply 6·224 by ·31831.

Exercise 7.

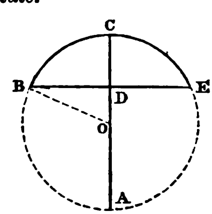
A mile being 5280 feet, the thousandth part of that, or 5·28 feet, will be the circumf. of the wheel. This we divide by 3·1416, or multiply by ·31831.

Problem VII.

Demonstration of the Rule.

From Euc. III, 35,

$$AD \times DC \doteq BD \times DE = BD^2.$$
$$\therefore AD \doteq BD^2 \div DC.$$



Exercise 1.

$$CD \doteq 1{\cdot}5\,;\; BD \doteq 5{\cdot}4\,;\; BD^2 \doteq 29{\cdot}16$$
$$\therefore AD \doteq 29{\cdot}16 \div 1{\cdot}5 = 19{\cdot}44.$$
$$AC \doteq 20{\cdot}94, \text{ and } OC = 10{\cdot}47.$$

* The Author has to thank Dr Rutherford for drawing his attention to that method, of which he was not aware when the "Complete Treatise" was published, and also for some corrections in the above note, as well as for the recent information it contains.

Exercise 2.

$$CD \doteq \tfrac{21}{4}\,;\ BE \doteq \tfrac{21}{4}\,;\ \text{and } BD \doteq \tfrac{21}{8}.$$
$$\therefore AD \doteq (\tfrac{21}{8})^2 \div \tfrac{21}{4} = \tfrac{21}{8} \times \tfrac{21}{8} \times \tfrac{4}{21} = \tfrac{21}{16} = 1{\cdot}3125.$$
$$AC \doteq 5{\cdot}25 + 1{\cdot}3125 = 6{\cdot}5625.$$

Exercise 3.

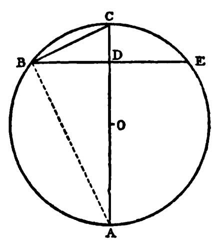
In $\triangle$ BDO we have BO = 100, and BD = 28.
$\therefore$, by Pr. I, OD = 96.
$CD \doteq 100 - 96 = 4.$

Exercise 4.

Here $CD \doteq 0{\cdot}7$, and $OC = 1$. $\therefore OD \doteq 0{\cdot}3$.
Then, in $\triangle$ BDO, $BO \doteq 1$, and $OD = 0{\cdot}3$.
$\therefore OD^2 \doteq 0{\cdot}09$; and $BD^2 \doteq 0{\cdot}91$.
$\therefore BD \doteq 0{\cdot}9539+$, and $BE = 1{\cdot}908-$.

Problem VIII.

Demonstration of the Rule.



Join AB. Then ABC being a right angle (Euc. III, 31), we have (from Euc. VI, 8, Cor.)

$$AC : BC :: BC : CD.$$
$$\therefore \text{(Euc. VI, 16) } AC \times CD \doteq BC^2,$$
$$\text{and } AC \doteq BC^2 \div CD.$$

Exercise 1.

$$BC \doteq 15 \text{ ft. } 7 \text{ in.} = 187 \text{ in.}$$
$$CD \doteq 7 \text{ ft. } 9\tfrac{1}{2} \text{ in.} = 93{\cdot}5 \text{ in.}$$
$$\therefore AC \doteq \frac{187 \times 187}{93{\cdot}5} = 187 \times 2.$$
$$\therefore OC \doteq 187 \text{ in.} = 15 \text{ ft. } 7 \text{ in.}$$

Exercise 2.

Here, diam. $AC \doteq 9$, and $CD = 4$.
$\therefore$, by 2d rev. For., $BC \doteq \sqrt{(9 \times 4)} = \sqrt{36}$.

Problem IX.

Demonstration of the Rule.

Since the circumf. of any circle may be regarded as consisting of 360 degrees, the given length of the circumference and 360° are merely the same thing in different kinds of measure, and so also are the length of the arc and the number of degrees it contains. Therefore the ratio of 360 degrees to the number of degrees in the arc, is identical with the ratio of the whole circumference to the arc, by whatever other measure they may be estimated.

Exercise 1.

As 360° : 57° 38′ :: 68 yards, or
As 21600′ : 3458′ :: 68 yds.; or, cancelling,
As 27 : 17·29 :: 17 yds. : 10 yds. 2·66 – ft.

Exercise 2.

As 360° : 1° :: 24,896 miles : $69{\cdot}1\dot{5}$ miles.

Problem X.

The rule requires no *demonstration.*

Exercise 1.

As 360° : 36° 42′ 16″ :: 25·1328; or
As 1296000″ : 132136″ :: 25·1328; or, cancelling,
As 27 : 16·517 :: 4·1888 : 2·5625 – .

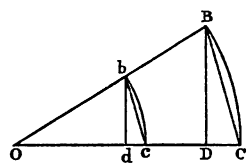
Problem XII.

The *natural sine* of an arc means the sine of an arc of the same number of degrees to a given radius, and is called so in contradistinction to the logarithmic sine, which is merely the logarithm of the natural sine. In like manner, the *natural versed sine* means the versed sine of an arc of the same number of degrees to some given radius.

The natural sine, versed sine, &c. corresponding to arcs

of the successive numbers of degrees and minutes, from 0° to 90°, have been computed and arranged in tables; and, for the sake of uniformity and simplicity, they are always calculated to the same radius, viz. 1. In the table at the end of the accompanying volume they are put down for every tenth minute in the quadrant; and in an adjacent column is the length of the corresponding arc to the same radius, 1.

Let BC be the arc whose length is required; OB or OC, its radius; BD, its sine; and DC, its versed-sine. Having set off Ob = 1, and, with that radius, described the arc bc, and drawn bd perp. to OC, bd will be the tabular natural sine, and dc the tabular natural versed-sine.



From the similar triangles BDC and bdc, we have

$$\text{(Euc. VI, 4) } OB : BD :: Ob : bd.$$

$$\therefore \text{(Euc. VI, 16) } OB \times bd \doteq Ob \times BD.$$

$$\text{Hence we have, } bd = \frac{Ob \times BD}{OB} = \frac{1 \times BD}{OB} = \frac{BD}{OB}.$$

That is to say, the tabular nat. sine, bd, is found by dividing the given sine BD by the given radius OB.

Having found bd, we look for it in its proper column in the table, and opposite it we find the length of the arc bc.

Then again we return from the arc bc to the arc BC; for Ob : OB :: *arc* bc : *arc* BC, and consequently,

$$Arc\ BC \doteq arc\ bc \times \frac{OB}{Ob} = arc\ bc \times \frac{OB}{1} = arc\ bc \times OB.$$

In order not to interrupt the demonstration we have assumed that Ob : OB :: *arc* bc : *arc* BC. It may be proved thus:—

If the whole circles were completed of which bc and BC are arcs, and if the circumferences of these two circles were named, respectively, 4 m and 4 *m*, m and *m* being their quadrants. Then the arcs m and *m* will be subtended by right angles at the centre (Euc. VI, 33).

2 Ob : 2 OB :: 4 m : 4 *m*, (Playfair's Geom. Sup. I, 6).

∴ Ob : OB :: m : *m*, (Euc. V, 15 and 11).

But m : *arc* bc :: ∟ : ∠ O, (Euc. VI, 33),

and *m* : *arc* BC :: ∟ : ∠ O, (Euc. VI, 33).

∴ m : *arc* bc :: *m* : *arc* BC,

and, alternately, m : *m* :: *arc* bc : *arc* BC.

$$\text{But } Ob : OB :: m : m.$$
$$\therefore Ob : OB :: \textit{arc } bc : \textit{arc } BC.$$

The whole of this process is necessary unless we assume more than is to be found in our ordinary editions of Euclid, but the substance of the latter part of the demonstration is simply this, that, since the circumferences of the two circles are to each other as their diameters, or as their radii, the arcs bc and BC, which are like parts of those circumferences, are in the same proportion.

Exercise 1.

Nat. sine $\doteq 21 \div 24 =$ ·875.
Nearest on Table, ·8746.
Corresponding tabular arc, 1·0647.
$1·0647 \times 24 \doteq 25·5528$.

Exercise 2.

Nat. versed-sine $\doteq 5 \div 10 = ·5$.
Corresponding tabular arc, 1·0472.
$1·0472 \times 10 \doteq 10·472$.

Exercise 3.

Chord of whole arc $\doteq 600$.
Sine of half arc $\doteq 300$. (See Note 2.)
Nat. sine of half arc $\doteq 300 \div 900 = ·3333 +$.
Nearest on Table, ·3338.
Corresponding arc, ·3403.
$·3403 \times 900 \times 2 \doteq 612·54$.

Exercise 4.

Nat. v. sine of half arc $\doteq 700 \div 725 = ·9655$.
Nearest on Table, ·9651.
Corresponding arc, 1·5359.
$1·5359 \times 725 \times 2 \doteq 2227·055$.

Exercise 5.

Here, by Pr. VII, diam. $\doteq 288 + 50 = 338$.
$\therefore$ radius $\doteq 169$.
N. sine of half arc $\doteq 120 \div 169 = ·7101 -$.
Nearest on Table, ·7092.
Corresponding arc, ·7883.
$·7883 \times 169 \times 2 \doteq 266·4454$.

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Exercise 6.

Height, by Pr. I, $\doteq \sqrt{(25^2 - 20^2)} = 15$.
Diameter, by Pr. VIII, $\doteq 25^2 \div 15 = \frac{125}{3}$.
Hence, radius $\doteq \frac{125}{6}$. Therefore
N. sine of half arc $\doteq 20 \div \frac{125}{6} = \frac{20}{1} \times \frac{6}{125} = \cdot96$.
Nearest on Table $\cdot9596$.
Corresponding arc, $1\cdot2857$.
$1\cdot2857 \times \frac{125}{6} \times \frac{2}{1} \doteq 53\cdot57+$.

Exercise 7.

Finding the supplemental arc, according to Note 1.

$$
\begin{array}{lr}
\text{Nat. sine} \doteq 1\cdot8 \div 9 = \cdot2. & \\
\text{Nearest on Table,} & \cdot1994. \\
\text{Corresponding arc,} & \cdot2007. \\
\text{Sup. arc} \doteq \cdot2007 \times 9 & = \ 1\cdot8063. \\
\text{Semicircumf.} \doteq 3\cdot1416 \times 9 & = \underline{28\cdot2744.} \\
\text{Arc required,} & 26\cdot4681.
\end{array}
$$

Exercise 8.

$$\text{Diameter} \doteq \frac{60 \times 60}{144} + 144 = 169.$$
$$\therefore \text{ radius} \doteq 84\cdot5.$$

Then, since the height of the arc is greater than the radius, the arc is greater than the semicircumference, and the half arc is greater than a quadrant. Therefore, according to Note 1, we first find the supplement of the half arc.

The sine of the half arc (by Note 2) is the half of 120, that is, 60, which is also the sine of its supplement. Therefore

$$
\begin{array}{lr}
\text{N. sine of sup.} \doteq 60 \div 84\cdot5 = \cdot7101-. & \\
\text{Nearest n. sine on Table,} & \cdot7092. \\
\text{Corresponding arc,} & \cdot7883. \\
\quad \text{Sup. arc} \doteq 0\cdot7883 \times 84\cdot5, & \\
\quad \text{Semicircumf.} \doteq \underline{3\cdot1416} \times 84\cdot5. & \\
\quad \therefore \text{ half arc} \doteq 2\cdot3533 \times 84\cdot5. & \\
\text{Whole arc} \doteq 2\cdot3533 \times 169 = 397\cdot7077. &
\end{array}
$$

Problem XIII.

Demonstration of the Rule.

Suppose we wish to find a rule for determining approximately the length of the arc from the chord of the arc and the chord of half the arc, putting a for the arc, we may, if we choose, assume the equation

$$a = Pc + Qc',$$

P and Q being co-efficients whose values are to be determined. Then $c \doteq 2 \sin \frac{1}{2} a$, and $c' \doteq 2 \sin \frac{1}{4} a$, the radius being that of the required arc.

$$\therefore a = Pc + Qc' \doteq 2 P \sin \tfrac{1}{2} a + 2 Q \sin \tfrac{1}{4} a.$$

But m being any arc, and r its radius, we have, by a well-known formula,*

$$\text{Sin } m = m - \frac{1}{6} \cdot \frac{m^3}{r^2} + \frac{1}{120} \cdot \frac{m^5}{r^4} - \&\text{c}.$$

If we insert successively $\frac{1}{2} a$ and $\frac{1}{4} a$ in the place of m, we have

$$\text{Sin } \tfrac{1}{2} a = \frac{a}{2} - \frac{1}{6} \cdot \frac{a^3}{8r^2} + \frac{1}{120} \cdot \frac{a^5}{32r^4} - \&\text{c., and}$$

$$\text{Sin } \tfrac{1}{4} a = \frac{a}{4} - \frac{1}{6} \cdot \frac{a^3}{64r^2} + \frac{1}{120} \cdot \frac{a^5}{1024r^4} - \&\text{c}.$$

$$\therefore 2 P \sin \tfrac{1}{2} a \doteq Pa - \frac{1}{6} \cdot \frac{P}{4} \cdot \frac{a^3}{r^2} + \frac{1}{120} \cdot \frac{P}{16} \cdot \frac{a^5}{r^4} - \&\text{c}.$$

$$\text{and } 2 Q \sin \tfrac{1}{4} a = \frac{Q}{2} a - \frac{1}{6} \cdot \frac{Q}{32} \cdot \frac{a^3}{r^2} + \frac{1}{120} \cdot \frac{Q}{512} \cdot \frac{a^5}{r^4} - \&\text{c}.$$

$$\therefore a = 2 P \sin \tfrac{1}{2} a + 2 Q \sin \tfrac{1}{4} a \doteq$$

$$\left(P + \frac{Q}{2}\right) a - \frac{1}{6}\left(\frac{P}{4} + \frac{Q}{32}\right)\frac{a^3}{r^2} + \frac{1}{120}\left(\frac{P}{16} + \frac{Q}{512}\right)\frac{a^5}{r^4} - \&\text{c}.$$

If, then, we could find such values of P and Q that $\left(P + \frac{Q}{2}\right)$ should be $= 1$, and all the other co-efficients, $\left(\frac{P}{4} + \frac{Q}{32}\right)$, $\left(\frac{P}{16} + \frac{Q}{512}\right)$, &c. severally $= 0$, we should have an identical equation, viz. $a = a$, and a perfect rule instead of an approximation. But it is impossible to assign any

* The demonstration of this formula may be found in the Key to the Author's "*Complete* Treatise," Part III, Pr. XI, or in any work on the higher Calculus.

such values to P and Q. All that we can do, is to find such values for them that we shall have

$$P+\frac{Q}{2}=1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and, } \frac{P}{4}+\frac{Q}{32}=0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and we shall then have our formula holding true in the first and second terms of the series, and erring only in the subsequent terms, which, we shall find, consist of small fractions, and are comparatively insignificant, especially if the arc is a small part of the circumference; that is, if a is small compared with r.

Resolving the two preceding equations to find the values of P and Q, we find $P=-\frac{1}{3}$ and $Q=+\frac{8}{3}$. Therefore returning to our assumed equation, viz. $a=Pc+Qc'$, and inserting for P and Q the values just found, we have

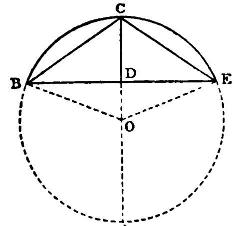
$$a=-\frac{1}{3}c+\frac{8}{3}c'=\frac{8c'-c}{3},$$

which, expressed in words, is our rule.

NOTE. To find a *general expression* for the *error* produced by this rule, see the Key to the Author's Complete Treatise on the same subject, Part III, Pr. XIII. It may be sufficient to say, here, that, when the arc is not of more than 60°, the error does not exceed the 6000th part of the whole, that, if the arc is short of 90° the error is less than 1 in 1300, and that if it comes up to 120°, the error is as much as 1 in 400. The deviation from the truth is insignificantly small for arcs of less than 30°, and so great as to render the rule entirely useless when the arc is nearly a semicircumference. The result, by this rule, always comes out less than it ought to be.

Exercise 3.

Here we have BD = 24·25,
and CD = 18·25.
∴ $BD^2 \doteq 588{\cdot}0625$,
and $CD^2 \doteq 333{\cdot}0625$.
∴ $BC^2 \doteq 921{\cdot}1250$,
and BC = 30·35 +.



Exercise 4.

$$OD \doteq \sqrt{(BO^2 - BD^2)} = \sqrt{(10^2 - 8^2)} = 6.$$
$$CD \doteq 10 - 6 = 4.$$
$$BC \doteq \sqrt{(BD^2 + CD^2)} = \sqrt{(8^2 + 4^2)} = 8{\cdot}9443 -.$$

Exercise 5.

$$BC \doteq \sqrt{(CD \times CA)} = \sqrt{(4 \times 30)} = 10.9545 -.$$
$$BD \doteq \sqrt{(CD \times DA)} = \sqrt{(4 \times 26)} = 10{\cdot}1980 +.$$
$$BE = 2\,BD \doteq 20{\cdot}3960 +.$$

Problem XIX.

Demonstration of the Rules.

Rule 1 depends on the assumption that the circumference of an ellipse is an arithmetical mean between that of a circle described upon the major axis and that of another circle described upon the minor axis, or, in other words, that the circumference of the ellipse is equal to the circumference of a circle described on a diameter which is as much less than the one axis as it is greater than the other, —an assumption which is not far from the truth when the axes are nearly equal, or when the ellipse approaches the circular form. But when the ellipse is very eccentric,— that is, when the axes are very unequal, the error is great, amounting in extreme cases to more than one-fifth of the whole, and making the result in all cases too small.

Rule 2 makes an assumption which is somewhat nearer the truth—that the circumference of an ellipse is equal to that of a circle described upon a diameter equal to the square root of half the sum of the squares of the two axes. This invariably gives the result too great. The error, as by the former rule, is trifling when the axes are nearly equal; but, in cases of the greatest eccentricity it is more than one-tenth of the whole.

These assumptions may, at first, have been merely hypothetical, verified by trials, but they may be made to rest on surer ground, and the amount of error may be ascertained in general terms, by deducing both rules from

a true expression for the value of the elliptic circumference. That expression is the following:—

$$\Pi \times (a+b) \times (1 + \tfrac{1}{4} c^2 + \tfrac{1}{64} c^4 + \tfrac{1}{256} c^6 + \&c.)^*$$

a and b being the semiaxes, and c being the quotient of $(a-b) \div (a+b)$.

Since $(a-b)$ must always be less than $(a+b)$, c is always less than unity, and c^2, c^4, &c. constantly diminish as the powers advance. Rule 1, therefore, expressed by the formula, circumf. $= \Pi \times (a+b)$, will nearly agree with the true series when c is a small fraction, or when a is not much greater than b. The error will be expressed in general terms by

$$\Pi \times (a+b) \times (\tfrac{1}{4} c^2 + \tfrac{1}{64} c^4 + \tfrac{1}{256} c^6 + \&c.).$$

Rule 2, expressed by the formula, circumf. $= \Pi \times \sqrt{(2a^2 + 2b^2)}$, gives a closer approximation; for

$$2a^2 + 2b^2 \doteq (a+b)^2 + (a-b)^2 = (a+b)^2 + (a+b)^2 \times \left(\frac{a-b}{a+b}\right)^2$$

$$= (a+b)^2 + (a+b)^2 \times c^2 = (a+b)^2 \times (1 + c^2).$$

$$\therefore \Pi \times \sqrt{(2a^2 + 2b^2)} \doteq \Pi\,(a+b) \times \sqrt{(1+c^2)}.$$

$$= \Pi\,(a+b) \times (1 + \tfrac{1}{2}c^2 - \tfrac{1}{8}c^4 + \tfrac{1}{16}c^6 - \&c.),$$

an expression which exceeds the true value given above by

$$\Pi \times (a+b) \times (\tfrac{1}{4}c^2 - \tfrac{9}{64}c^4 + \tfrac{15}{256}c^6 - \&c.).$$

Exercise.

By Rule 2. $\tfrac{1}{2}(3^2 + 2^2) \doteq \tfrac{1}{2}$ of $13 = 6{\cdot}5$.

$\sqrt{6{\cdot}5} \doteq 2{\cdot}5495 +$.

$2{\cdot}5495 \times 3{\cdot}1416 \doteq 8{\cdot}0095 +$.

PROBLEM XXVI.

Demonstration of the Rule.

The peculiar mode of demonstrating the rule for this Problem will depend on the definitions given to the words "similar figures" and "corresponding lines, or lineal dimensions": but, in all cases of *rectilineal* figures (or of *any* figures, when the two dimensions are both straight lines), the demonstration will be very simple if, in each

* This expression for the circumference of an ellipse was first given by Sir James Ivory in the Edinburgh Philosophical Transactions, Vol. IV. The demonstration may be found, expanded and simplified, in the Key to the Author's Complete Treatise, Part III, Problem XIX.

figure, we connect the lines, which the two dimensions express, by one or more triangles. Then, (by Euc. v, 12) it may be shown that the same property applies not only to single straight lines, but to lines made up of any number of corresponding straight lines, such as the circumferences of similar polygons. This being proved, the same property is next extended to curvilineal circumferences, by showing that it holds good in the case of the circumferences of similar polygons inscribed within them, whatever their number of sides, and however close their circumferences approach to those of the curvilineal circumferences. The demonstration is applied in the same way to curve lines which are *not complete* circumferences.

Exercise 1.

As 60 feet : 5 inches :: 25 feet : $2\frac{1}{12}$ inches.

Exercise 2.

As 5 inches : 60 feet :: 3·75 inches : 45 feet.

Exercise 3.

As 72 inches : 142 inches :: $11\frac{1}{2}$ inches : $22\frac{49}{72}$ inches.

Exercise 4.

As 3·8 inches : 103 miles :: 11·44 inches : 310 + miles.

Exercise 5.

As 1000 : 136 in. :: 866 : $117\frac{3}{4}$ + in., or
As 1000 : 866 :: 11 ft. 4 in. : 9 ft. $9\frac{3}{4}$ in.

Exercise 6.

As 22 : 12 :: 7 : $3\frac{9}{11}$.
Since 22 is not the exact number, neither will $3\frac{9}{11}$.

Exercise 7.

As 1·3066 : 1·2071 :: 6 : 5·543 +.

Exercise 8.

As 1 mile : 600 miles :: $\frac{1}{40}$ in. : 15 inches.
Allowance for margin $\doteq 2\frac{1}{2} \times 2 = 5$
Ans. 20 inches.

Exercise 9.

As 126 feet : 84 feet :: 31 feet : 20 ft. 8 in.

Exercise 11.

As 9·86 feet : 132 feet :: 12 feet : 160·65 – feet.

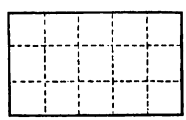
CHAPTER IV.

MENSURATION OF SURFACES.

PROBLEM I.

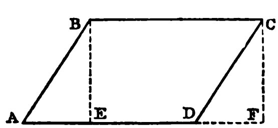
Demonstration of the Rule.

In the case of the rectangle, let the longer side be divided into as many parts as there are units in the length, and the shorter side into as many parts as there are units in the breadth. Then, through the points of division let lines be drawn parallel to the sides of the rectangle. These will divide the rectangle into a number of squares, each of which is a square unit of the same denomination as the units of the length and breadth. Counting these squares then in rows, each row extending the whole length of the rectangle, we have as many rows as there are units in the breadth, and as many squares in each row as there are units in the length. Therefore the whole number of squares will be obtained by multiplying one of those numbers by the other. Thus if the length is 5 and the breadth 3, we shall have 3 rows of 5 squares each, that is, 3 times 5.



The square is merely a particular case of the rectangle.

The oblique-angled parallelogram may easily be shown to be equal to a rectangle of the same length and breadth, for the former would be converted into the latter by taking a right-angled triangle from the one end and adding it to the other. Thus, in the annexed diagram, the ◺ AEB is evidently equal to the ◺ DFC; and therefore the parallelogram ABCD is equal to the rectangle BCFE, and



both have the same length, BC, and the same breadth, BE. Consequently their areas will be found by the same process.

Exercise 3.

$$7\tfrac{1}{8} \times 5\tfrac{1}{3} \doteq \tfrac{57}{8} \times \tfrac{16}{3} = \tfrac{19}{1} \times \tfrac{2}{1} = 38.$$

Exercise 5.

$$26\tfrac{1}{2} \times 12 \times 11\tfrac{3}{4} \doteq \tfrac{53}{2} \times \tfrac{12}{1} \times \tfrac{47}{4} = \tfrac{53}{2} \times \tfrac{3}{1} \times \tfrac{47}{1}$$
$$= \tfrac{7473}{2} = 3736\tfrac{1}{2} \text{ sq. inches.}$$

This result must be divided successively by 12 and by 12 to reduce it into parts and then into feet.

The same result might have been obtained even more easily by duodecimals, thus:

Ft.	in.	
26	6	
0	$11\frac{3}{4}$	
24	3	6
1	1	3
	6	$7\frac{1}{2}$
25	11	$4\frac{1}{2}$.

Exercise 6.

$$944 \times 587 \doteq 554128 \text{ sq. links} = 5{\cdot}54128 \text{ ac.}$$

Exercise 7.

$$9)\underline{555025} \text{ sq. ft.} = 745^2.$$
$$61670 - \text{sq. yards.}$$
$$61670 \div 4840 \doteq 12{\cdot}742 - \text{acres.}$$
$$555025 \div 330 \doteq 1682 - \text{cottages.}$$

Exercise 8.

$$84 \times 47\tfrac{3}{4} \doteq 4011 \text{ sq. feet} = 445\tfrac{2}{3} \text{ sq. yds.}$$

The rest may be done either by Practice or Proportion.

Exercise 9.

$$(7\tfrac{5}{8})^2 = (\tfrac{61}{8})^2 = \tfrac{3721}{64} \doteq \text{area of one side.}$$
$$\tfrac{3721}{64} \times 6 = 348\tfrac{27}{32} \doteq \text{whole area.}$$

Exercise 10.

	Ft.	in.	ft.	ft.	pts.
Top	$\doteq 2$	: 6	$\times 3$	= 7 :	6
Side	$\doteq 1$	: 4	$\times 3$	= 4 :	0
End	$\doteq 1$	: 4	$\times 2\frac{1}{2}$	= 3 :	4
				14 :	10
					2
			Ans.	29 :	8.

Exercise 18.

$\sqrt{4830{\cdot}25} \doteq 69{\cdot}5.$

Exercise 19.

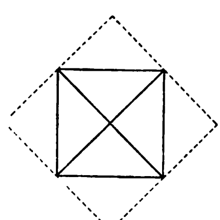
70 ft. 10 pts. 1 in. $\doteq$ 10201 sq. inches.
$\sqrt{10201} \doteq 101$ inches.

Exercise 21.

$\sqrt{4840} \doteq 69{\cdot}6 -.$

Problem II.

By referring to the diagram, it will be seen that the square with the dotted outline is double of the inside square. But the side of the outer square is equal to the diagonal of the inner square. Therefore the area of the inner square is half the square of its diagonal.



We derive the same inference from Problem I of the last chapter. For the square, whose area is required, being divided into two right-angled triangles by its diagonal, we know that in either of these, the square of the diagonal is equal to twice the square of the side, since the two sides are equal.

$$\therefore \text{ required square} = \text{side}^2 \doteq \tfrac{1}{2}\text{ diag}^2.$$

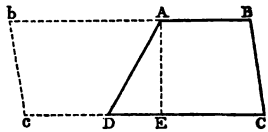
Exercise 3.

By the reverse formula,

$$\text{Diag.} \doteq \sqrt{(2 \times 578)} = \sqrt{1156} = 34.$$

Problem III.

Demonstration of the Rule.



Produce the two parallel sides till $Ab \doteq CD$, and $cD = AB$. Then we shall have $Bb = Cc$. Therefore, (Euc. I, 33) bc is equal and parallel to BC, and the figure BbcC is a parallelogram, the area of which, (by Pr. I) $\doteq Cc \times AE \doteq (S+S') \times B$.

But the trapezoid is evidently half the parallelogram. Therefore area of trapezoid $\doteq \frac{1}{2}(S+S') \times B$, or half the sum of the two parallel sides multiplied by the perpendicular breadth.

Exercise 4.

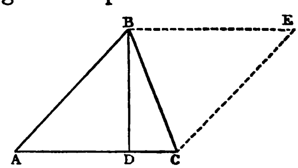
$$3{\cdot}48 + 8{\cdot}31 \doteq 11{\cdot}79.$$

$$11{\cdot}79 \times 1{\cdot}38 \doteq 16{\cdot}2702 \text{ sq. ch.} = 1{\cdot}62702 \text{ ac.}$$

Problem IV.

Demonstration of the Rule.

The lines BE and CE being drawn parallel to AC and AB, two sides of the Δ ABC, ABEC is a parallelogram whose area (by Problem I) $\doteq AC \times BD = B \times P$.



But the Δ ABC is half the parallelogram (Euc. I, 34).

$$\therefore \text{Area of } \Delta \text{ ABC} \doteq \tfrac{1}{2}(B \times P).$$

Exercise 2.

$$29\tfrac{1}{2} \times 33\tfrac{2}{3} \div 2 \doteq \tfrac{59}{2} \times \tfrac{101}{3} \times \tfrac{1}{2} = \tfrac{5959}{12} = 496\tfrac{7}{12}.$$

Exercise 3.

5 ft. 3 in. $\times$ 8 in. $\doteq$ 4 ft. 4 pts. 6 in.

Exercise 4.

By Ch. III, Pr. I, Perp. $\doteq \sqrt{(205^2 - 200^2)} = 45$.

Exercise 5.

Here AB $\doteq$ BC, and AD $=$ DC.

Now, in $\triangle$ ABD, we have AB $= 572$ in., and AD $= 220$ in. Hence (by Pr. I of Ch. III) BD $\doteq 528$ in.

$$528 \times 220 \doteq 116160 \text{ sq. in.} = 806\tfrac{2}{3} \text{ sq. ft.}$$

Exercise 6.

In this instance, again, AB $\doteq$ BC, and AD $=$ DC, and the three sides, AB, BC, and AC, are each 3·4. $\therefore$ in $\triangle$ ADB, we have AB $= 3{\cdot}4$, and AD $= \frac{1}{2}$ AC $= 1{\cdot}7$. Hence BD $= \sqrt{(AB^2 - AD^2)} \doteq \sqrt{(3{\cdot}4^2 - 1{\cdot}7^2)} = \sqrt{5{\cdot}67} = 2{\cdot}9445 -$.

Exercise 7.

By drawing lines from the centre of the hexagon to each of the angular points, we divide it into six equal equilateral triangles, having each side $= 1$.

In any one of these we find the altitude as in Ex. 6, viz.—

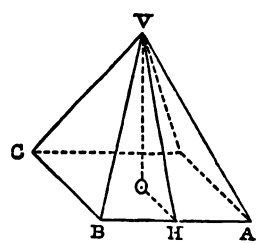
$$P = \sqrt{(1^2 - {\cdot}5^2)} = \sqrt{{\cdot}75} = {\cdot}8660254.$$

$$\therefore \text{ Area of one } \triangle \doteq {\cdot}4330127,$$

$$\text{and area of hexagon} \doteq {\cdot}4330127 \times 6.$$

Exercise 8.

Each side of the pyramid is a triangle, as BVA, whose altitude is VH. To find VH we have recourse to the triangle VOH, which is right-angled at O. In this, HO $= \frac{1}{2}$ BC $\doteq$ 17·5, and VO $\doteq$ 42. $\therefore$ HV $\doteq$ 45·5 (by Pr. I of Ch. III).



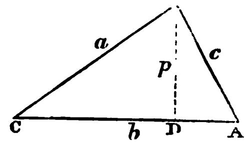
$$\therefore \text{ Area of } \triangle \text{ BVA} \doteq \tfrac{1}{2} \text{ of } 45{\cdot}5 \times 35 = 796{\cdot}25$$

	4
Area of four sides, ...	3185
Area of base, ...	1225
Ans.	4410.

Problem V.

Demonstration of the Rule.

Let ABC be any triangle, of which the longest side, AC, is taken as the base, a perp., BD, being drawn to it from the vertex. Let a, b, and c represent, respectively, the given sides opposite the angles A, B, and C; and let CD and DA, which are not given, be expressed by the letters x and y. We have (by Euc. I, 47),



$$\text{in} \triangle \text{ADB, BD}^2 = c^2 - y^2,$$

$$\text{and in} \triangle \text{CDB, BD}^2 \doteq a^2 - x^2.$$

$$\therefore c^2 - y^2 \doteq a^2 - x^2;$$

$$\text{or, transposing, } x^2 - y^2 \doteq a^2 - c^2.$$

$$\text{But } x + y \doteq b.$$

Dividing one of these two equations by the other,

$$x - y \doteq \frac{a^2 - c^2}{b}.$$

$$\text{But } x + y \doteq b = \frac{b^2}{b}.$$

Adding these two equations together,

$$2x \doteq \frac{a^2 + b^2 - c^2}{b},$$

$$\text{and } x \doteq \frac{a^2 + b^2 - c^2}{2b} = \text{CD}.$$

$$\therefore \text{CD}^2 \doteq \frac{a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4}{4b^2}.$$

$$\text{CB}^2 \doteq a^2 = \frac{4a^2b^2}{4b^2}.$$

$$\therefore \text{BD}^2 = \text{CB}^2 - \text{CD}^2 \doteq \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{4b^2};$$

$$\text{and BD} \doteq \frac{1}{2b} \times \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}.$$

$$\text{But, area of } \triangle \text{ABC} \doteq \tfrac{1}{2}(\text{AB} \times \text{BD})$$

$$= \tfrac{1}{4}\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)},$$

$$= \sqrt{\left(\frac{a+b+c}{2} \times \frac{a+b-c}{2} \times \frac{a-b+c}{2} \times \frac{-a+b+c}{2}\right)},$$

an expression which is identical with our rule, since the first factor is half the sum of the three sides, and each of *the others is* that half sum diminished by one of the sides.

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NOTE. The reason of selecting the *longest* side as the base, is to avoid a slight variation which would be necessary in the demonstration, if either of the angles A or C were obtuse.

Exercise 1.

$$21 \times 6 \times 7 \times 8 \doteq 7056.$$

Exercise 2.

$$3162 \times 737 \times 744 \times 1681 \doteq 2{,}914{,}539{,}881{,}616.$$
$$\sqrt{2{,}914{,}539{,}881{,}616} \doteq 1{,}707{,}202 + \text{sq. links.}$$

Exercise 3.

$$68{\cdot}5 \times 33{\cdot}5 \times 17{\cdot}5 \times 17{\cdot}5 = 702767.$$
$$\sqrt{702767} \doteq 838{\cdot}3 +.$$

Exercise 4.

$$\sqrt{(35 \times 15 \times 14 \times 6)} \doteq \sqrt{44100} = 210.$$
Or, $\sqrt{(35 \times 15 \times 14 \times 6)} \doteq \sqrt{(7 \times 5 \times 5 \times 3 \times 7 \times 2 \times 3 \times 2)}$
$$= \sqrt{(7^2 \times 5^2 \times 3^2 \times 2^2)} = 7 \times 5 \times 3 \times 2 = 210.$$

Exercise 5.

$$1194 \times 398 \times 398 \times 398 \doteq 75{,}275{,}481{,}648.$$
$$\sqrt{75{,}275{,}481{,}648} \doteq 274364 -.$$
$$274364 \div (9 \times 4840) \doteq 6{\cdot}2985+.$$

Or thus:—

$$\sqrt{(1194 \times 398 \times 398 \times 398)} \doteq \sqrt{(3 \times 398^2 \times 398^2)}$$
$$= \sqrt{3} \times 398^2 = 1{\cdot}73205 \times 158404 = 274364 -.$$

Exercise 6.

$$\text{Base} \doteq \sqrt{(2{\cdot}175 \times {\cdot}725 \times {\cdot}725 \times .725)}$$
$$= \sqrt{{\cdot}828845} = 0{\cdot}9104$$
$$3 \text{ Sides} \doteq 3\sqrt{(3{\cdot}405 \times 1{\cdot}955 \times {\cdot}725 \times {\cdot}725)}$$
$$= 3\sqrt{3{\cdot}4989} = \underline{5{\cdot}6116}$$
$$\underline{\underline{6{\cdot}5220.}}$$

PROBLEM VI.

The rule requires no *demonstration.*

Exercise 1.

$$\triangle\ ABC \doteq 278 \times 264 = 73392$$
$$\triangle\ ADC \doteq 278 \times 235 = \underline{65330}$$
$$\text{Quadrilateral ABCD} \doteq \underline{\underline{138722}} \text{ sq. l.}$$

Or thus:—

$$\text{Quad. ABCD} \doteq 278 \times (264 + 235) = 278 \times 499.$$

Exercise 2.

Converting at once the links into chains,

$$9{\cdot}635 \times 3{\cdot}355 \times 4{\cdot}995 \times 1{\cdot}285 \doteq 207{\cdot}483 +.$$
$$10{\cdot}125 \times 2{\cdot}795 \times 5{\cdot}555 \times 1{\cdot}775 \doteq 279{\cdot}035 +.$$
$$\triangle\ ABC \doteq \sqrt{207{\cdot}483} = 14{\cdot}404 + \text{ sq. chains.}$$
$$\triangle ADC \doteq \sqrt{279{\cdot}035} = \underline{16{\cdot}704} + \ldots\ldots\ldots\ldots$$
$$\text{Trapezium ABCD} \doteq \underline{\underline{31{\cdot}108}} + \ldots\ldots\ldots\ldots$$

PROBLEM VII.

Demonstration of the Rule.

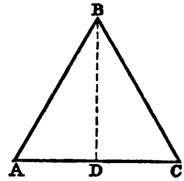
In the case of the square:—

$$\text{Area of square} \doteq L^2 = L^2 \times 1,$$

which agrees with our rule in this case, since the tabular multiplier for the area is 1·0000.

In the case of the equilateral triangle:—

$$\begin{aligned} BD &\doteq \sqrt{(AB^2 - AD^2)} \\ &= \sqrt{(L^2 - \tfrac{1}{4} L^2)} \\ &= \sqrt{(\tfrac{3}{4} L^2)} \\ &= L \times \tfrac{1}{2} \sqrt{3}. \end{aligned}$$



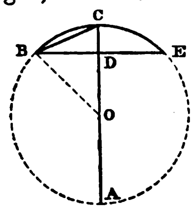
$$\text{Area of } \triangle\ ABC \doteq \tfrac{1}{2}(AC \times BD) = \tfrac{1}{2}(L \times L.\ \tfrac{1}{2}\sqrt{3})$$
$$= L^2 \times \tfrac{1}{4}\sqrt{3} = L^2 \times \tfrac{1}{4} \text{ of } 1{\cdot}73205 + = L^2 \times 0{\cdot}43301 +.$$

In the case of the hexagon :—

By drawing lines from the centre to each of the angular points, it will be seen that the hexagon is divided into six equilateral triangles, the area of each of which, we have just seen, will be expressed by $L^2 \times \frac{1}{4}\sqrt{3}$. Therefore the area of the hexagon $\doteq 6 \times L^2 \times \frac{1}{4}\sqrt{3} = L^2 \times \frac{3}{2}\sqrt{3} = L^2 \times \frac{3}{2}$ of $1{\cdot}73205 = L^2 \times 2{\cdot}5981-$.

In the case of the dodecagon :—

In the annexed figure let BC represent the side of a dodecagon, BE being the side of a hexagon, both inscribed in the circle. Then if R be put for the radius of the circle,



$$BD \doteq \tfrac{1}{2}R, \text{ and}$$
$$OD^2 \doteq OB^2 - BD^2 = R^2 - (\tfrac{1}{2}R)^2 = \tfrac{3}{4}R^2.$$

$$\therefore OD \doteq R \times \tfrac{1}{2}\sqrt{3}, \text{ and}$$
$$CD = OC - OD \doteq R - R.\ \tfrac{1}{2}\sqrt{3} = R \times (1 - \tfrac{1}{2}\sqrt{3}).$$

Then, as was demonstrated under Problem VIII of the last chapter,

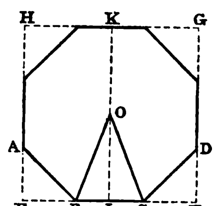
$$L^2 = BC^2 \doteq AC \times CD = 2R \times R.(1-\tfrac{1}{2}\sqrt{3}) = R^2 \times (2-\sqrt{3}).$$
$$\therefore R^2 \doteq \frac{L^2}{2-\sqrt{3}} = \frac{L^2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = L^2 \times \frac{2+\sqrt{3}}{4-3}$$
$$= L^2 \times (2+\sqrt{3}).$$
$$\text{Dodec.} \doteq 12\,\Delta\,BOC = 6\,(OC \times BD) = 3OC^2 = 3R^2$$
$$= L^2 \times (6 + 3\sqrt{3}) = L^2 \times 11{\cdot}1962-.$$

COR. Since dodec. $\doteq 3R^2$, the area of any regular dodecagon is equal to 3 times the square of the radius of the circumscribed circle.

In the case of the octagon :—

Having produced the alternate sides so as to form a square; having found the centre O; and having drawn the other lines in the diagram,

$$\text{Octagon} \doteq 8\,\Delta\,BOC$$
$$= 4\,(BC \times OI)$$
$$= 2\,BC \times IK.$$



We have now to find IK = EH = EF.

But EF $\doteq$ BC + EB + CF = BC + 2EB.

Now, in $\triangle$ AEB, which is evidently isosceles, we have $AB^2 = AE^2 + EB^2 = 2EB^2$.

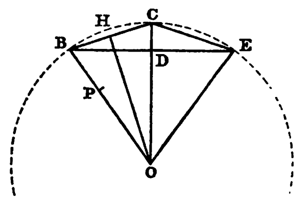
$$\therefore 4EB^2 \doteq 2AB^2.$$
$$\therefore 2EB \doteq AB \times \sqrt{2} = L.\sqrt{2}.$$
$$\therefore EF, \text{ or } IK, \doteq L + L.\sqrt{2} = L \times (1+\sqrt{2}), \text{ and}$$
$$\text{Octagon} = 2BC \times IK, \doteq 2L \times L \times (1+\sqrt{2}) = L^2 \times (2+2\sqrt{2})$$
$$= L^2 \times 4{\cdot}8284 +.$$

The same result may be obtained by observing that

$$\text{Octagon} \doteq \text{square } EFGH - 4\triangle AEB = EF^2 - 2EB^2$$
$$= EF^2 - AB^2 = L^2 \times (1+\sqrt{2})^2 - L^2 = L^2 \times (3+2\sqrt{2}) - L^2$$
$$= L^2 \times (2+2\sqrt{2}) = L^2 \times 4{\cdot}8284 +.$$

In the case of the pentagon :—

If OB, the radius of any circle, be divided in the point P, so that $OB \times BP \doteq OP^2$; if the chords BC and CE be placed in the circle, each = OP; and if OC and BE be joined; BC and BE will be the sides of a decagon and pentagon inscribed in the circle.



For we may prove, as in Euc. IV, 10, that, in the $\triangle$ BOC, the two angles at B and C are each double of the angle at O. Consequently these three angles are together equal to five times the angle at O. But these three angles are together equal to two right angles. Therefore the $\angle$ BOC $\doteq$ one-fifth of two right angles = one-tenth of four right angles, or one-tenth of all the angular space about the point O. But equal angles at the centre are subtended by equal arcs. Therefore the arc BC is one-tenth of the circumference, and the arc BE, = 2BC, is one-fifth of the circumference. Hence BC and BE are the sides of a decagon and pentagon inscribed in the circle.

Now, to find the numerical relation which BC or OP bears to BO, represent BO by the letter R, and BC or OP by x, and consequently BP by $R-x$.

$$\text{Then, } R \times (R-x) = R^2 - Rx \doteq x^2.$$

Resolving this quadratic equation, we find

$$BC, \text{ or } x, = R \times \tfrac{1}{2}(\sqrt{5}-1).$$
$$\therefore BC^2 \doteq R^2 \times \tfrac{1}{2}(3-\sqrt{5}).$$

Through O draw OH perp. to BC and consequently bisecting the line BC and the angle BOC. Then BH $\doteq$ $\frac{1}{2}BC$, and

$$BH^2 = \tfrac{1}{4}BC^2 \doteq R^2 \times \tfrac{1}{8}(3-\sqrt{5}).$$
$$OH^2 = BO^2 - BH^2 \doteq R^2 - R^2 . \tfrac{1}{8}(3-\sqrt{5})$$
$$= R^2 \times (1 - \frac{3-\sqrt{5}}{8}) = R^2 \times \frac{5+\sqrt{5}}{8}.$$

But $\angle CBE \doteq \tfrac{1}{2} \angle COE$ (Euc. III, 20)
$= \tfrac{1}{2} \angle BOC = \angle BOH.$

Therefore the two triangles BOH and CBD are equiangular, and consequently (Euc. VI, 4) have the sides about the equal angles proportional.

$$\therefore BO : OH :: CB : BD,$$

and (Euc. VI, 22) $BO^2 : OH^2 :: CB^2 : BD^2$,

$$\text{or, } R^2 : R^2 \times \frac{5+\sqrt{5}}{8} :: R^2 \times \frac{3-\sqrt{5}}{2} : BD^2.$$

$$\therefore BD^2 \doteq R^2 \times \frac{3-\sqrt{5}}{2} \times \frac{5+\sqrt{5}}{8} = R^2 \times \frac{5-\sqrt{5}}{8}.$$

But $BD \doteq \tfrac{1}{2} BE = \tfrac{1}{2} L$; and $BD^2 \doteq \tfrac{1}{4} L^2$.

$$\therefore \tfrac{1}{4}L^2 \doteq R^2 \times \frac{5-\sqrt{5}}{8}, \text{ or } L^2 \doteq R^2 \times \frac{5-\sqrt{5}}{2}.$$

$$\therefore BO^2 = R^2 \doteq$$

$$L^2 \times \frac{2}{5-\sqrt{5}} = L^2 \times \frac{2}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = L^2 \times \frac{5+\sqrt{5}}{10}.$$

$$OD^2 = BO^2 - BD^2 \doteq$$

$$L^2 \times \frac{5+\sqrt{5}}{10} - L^2 \times \tfrac{1}{4} = L^2 \times \left(\frac{5+\sqrt{5}}{10} - \tfrac{1}{4}\right)$$

$$= L^2 \times \frac{5+2\sqrt{5}}{20} = L^2 \times \tfrac{1}{4}(1 + \tfrac{2}{5}\sqrt{5}).$$

$$\therefore BD^2 \times OD^2 \doteq \tfrac{1}{4} L^2 \times L^2 . \tfrac{1}{4}(1+\tfrac{2}{5}\sqrt{5}) = L^4 \times \tfrac{1}{16}(1+\tfrac{2}{5}\sqrt{5}).$$

$$\therefore \triangle BOE = BD \times OD \doteq L^2 \times \tfrac{1}{4}\sqrt{(1+\tfrac{2}{5}\sqrt{5})}, \text{ and}$$

$$\text{Pentagon} = 5 \triangle BOE \doteq L^2 \times \tfrac{5}{4}\sqrt{(1+\tfrac{2}{5}\sqrt{5})}$$
$$= L^2 \times \tfrac{5}{4}\sqrt{1{\cdot}894427}$$
$$= L^2 \times \tfrac{5}{4} \text{ of } 1{\cdot}37638 = L^2 \times 1{\cdot}7205 - .$$

In the case of the decagon :—

BE, in the preceding diagram, being the side of a pentagon, and BC the side of a decagon inscribed in the circle, we have already named BE by the letter L; but now BC comes to be represented by that letter: we therefore, to avoid confounding the two, place an accent over the latter, making $BC = {}'L$.

It has just been proved that

$$BC^2 = {}'L^2 \doteq R^2 \times \tfrac{1}{2}(3-\sqrt{5}).$$

$$\therefore R^2 \doteq {}'L^2 \times \frac{2}{3-\sqrt{5}} = {}'L^2 \times \frac{2}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$
$$= {}'L^2 \times \frac{3+\sqrt{5}}{2}.$$

It has also been shown that

$$OH^2 \doteq R^2 \times \frac{5+\sqrt{5}}{8}.$$

$$\therefore OH^2 \doteq {}'L^2 \times \frac{3+\sqrt{5}}{2} \times \frac{5+\sqrt{5}}{8} = {}'L^2 \times \frac{5+2\sqrt{5}}{4}.$$

$$\therefore BC^2 \times OH^2 \doteq {}'L^4 \times \frac{5+2\sqrt{5}}{4}; \text{ and}$$

$$BC \times OH \doteq {}'L^2 \times \tfrac{1}{2}\sqrt{(5+2\sqrt{5})}.$$

$$\therefore \text{Decagon} = 10 \triangle BOC = 10 \times \frac{BC \times OH}{2} = 5 \times BC \times OH$$
$$\doteq {}'L^2 \times \tfrac{5}{2}\sqrt{(5+2\sqrt{5})} = {}'L^2 \times \tfrac{5}{2}\sqrt{9{\cdot}472136}$$
$$= {}'L^2 \times \tfrac{5}{2} \text{ of } 3{\cdot}07768 + = {}'L^2 \times 7{\cdot}6942 +.$$

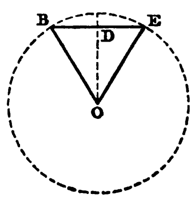
In the case of the heptagon :

BE being taken as the side of the heptagon, we have the area of the heptagon = 7 times $\triangle$ BOE = $\frac{7}{2}$(BE × OD).

But we have, in ⊾ ODB (Part III, Ch. VII, Pr. IV),

Radius : tan B :: BD = $\frac{1}{2}L$: OD.

Or, using the natural tangent to radius 1,



1 : tan B :: $\frac{1}{2}L$: OD.

$\therefore$ OD $\doteq \frac{1}{2}L \times$ tan B, and

Heptagon = $\frac{7}{2}$(BE × OD)

$\doteq L^2 \times \frac{7}{4}$ tan B.

But $\angle$ BOE $\doteq \frac{1}{7}$ of 360° ;

and $\therefore \angle$ BOD $\doteq \frac{1}{14}$ of 360° = $\frac{2}{7}$ of 90°.

$\therefore \angle$ B $\doteq \frac{5}{7}$ of 90° = 64° 17$\frac{1}{7}$′.

The tangent of this angle taken from a table of natural tangents, is 2·07652 +.

$\therefore$ Hept. $\doteq L^2 \times \frac{7}{4}$ of 2·07652 = $L^2 \times$ 3·6339 +.

In the case of the undecagon :—

We find, in the same manner as with the heptagon, that

OD $\doteq \frac{1}{2} L \times$ tan B, and $\therefore$

Undecagon $\doteq L^2 \times \frac{11}{4}$ tan B.

But $\angle$ BOD $\doteq \frac{1}{22}$ of 360° = $\frac{2}{11}$ of 90°.

$\therefore \angle$ OBD $\doteq \frac{9}{11}$ of 90° = 73° 38$\frac{2}{11}$′.

Tan B $\doteq$ 3·40569 –.

$\therefore$ Undecagon $\doteq L^2 \times$ 9·3656 +.

Exercise 2.

$$\text{Area} \doteq \frac{24336 \times 4{\cdot}8284}{9 \times 4840} = \frac{270{\cdot}4 \times 1{\cdot}2071}{121} \text{ ac.}$$

Exercise 3.

$$\begin{aligned}\text{Area} &\doteq 389^2 \times {\cdot}43301 = 151321 \times {\cdot}43301 \\ &= 65524 - \text{sq. links} = {\cdot}65524 - \text{ac.}\end{aligned}$$

Exercise 4.

The area of each end, by the rule, is 5·3128. The sides are rectangles, the area of one of which is $8{\cdot}5 \times 1{\cdot}43$; or the area of all the six is $51 \times 1{\cdot}43 = 72{\cdot}93$.

Exercise 5.

The area of the base, by the rule, is $45^2 \times 3{\cdot}6339 = 7359 -$ sq. inches.

The area of each side is $45 \times 32 = 1440$ sq. in.

Problem VIII.

The rule needs no *demonstration*.

Exercise 1.

Field ABCDH.		Field DEFGH.	
△ ABC	≐ 33·211	△ DEF	≐ 46·275
△ ACD	≐ 77·552	△ DFH	≐ 88·845
△ ADH	≐ 76·056	△ HFG	≐ 99·763
Total,	186·819 sq. ch.	Total,	234·883 sq. ch.

Exercise 2.

△ ABG	≐ 88,707
△ GBF	≐ 202 330
◺ BIC	≐ 10 537
Trap. CIKD	≐ 5 824
Trap. DKLE	≐ 13 728
◺ ELF	≐ 14 110
Total, ...	335.236 sq. links.

PROBLEM XII.

Geometrical Demonstration of the Rule.

Let a regular polygon of any number of sides be described about the circle, st. lines being drawn from all its angular points to the centre. These lines will divide the polygon into as many triangles as it has sides. If, then, we express the side of the polygon by the letter L, the number of sides by N, and the radius of the circle by R, the whole perimeter of the polygon will be $L \times N$; the area of each of the triangles will be $\frac{1}{2}$ (L × R); and the area of the whole polygon, $\frac{1}{2}(L \times N \times R)$.

Now, if we continually increase the number of sides of the polygon, its perimeter approaches nearer and nearer to that of the circle; so that, if we suppose the number of sides to become infinite, we may consider the polygon as coinciding with the circle, and the expression, Polygon = $\frac{1}{2}(L \times N \times R)$, becomes $\frac{1}{2}(C \times R)$.

This is the same, in effect, as regarding the circle in the light of a polygon of an infinite number of sides, and applying to it the general rule for polygons. This simple view of the subject is undoubtedly sufficient to convince the understanding; but if more rigorous demonstration is called for, we must first show that, "About a circle we may describe a polygon of so many sides that its circumference shall differ from that of the circle by a quantity less than any assignable quantity, and that its area shall also differ from the area of the circle by a quantity less than any that can be assigned:" but into proof of that it is unnecessary to enter here, since demonstrations may be found in any of our modern treatises on Geometry, to which the subject more properly belongs. (See, for instance, *Playfair's Elements, Sup. B.* I, *Pr.* 4 *and* 5;—*Library of Useful Knowledge, Geometry, B.* III, § 4, *Pr.* 31 *and* 32,—*Elements de Geometrie, par Legendre, Liv.* IV, *Pr.* 12.—*Thomson's Euclid, Ap.* I, 39.)

None of these demonstrations, however, can be regarded as strictly perfect, since all assume, as an axiom, expressed or implied, that the circumference of a polygon is always greater than that of its inscribed circle.

Algebraical Demonstration.

Let z be an arc of a circle, and s the corresponding

sector, both regarded as variable, the radius being constant. Then

$$\partial s \doteq \tfrac{1}{2} r \partial z.$$

$\therefore$, by integrating, $s \doteq \frac{1}{2} rz$.

And, when z becomes the whole circumf., s becomes the area of the whole circle, $\therefore$

$$\text{Area of circle} \doteq \tfrac{1}{2}(R \times C).$$

It may be doubted if this or any other demonstration is superior in strictness to the simple one with which we commenced. They will all, it is believed, if traced to their various sources, be found to involve some consideration of the same subtle and undemonstrated kind as that involved in the first, although, in every case, quite satisfactory for conviction.

Exercise 2.

$$29{\cdot}5 \times 185{\cdot}35 \div 2 \doteq 2734 - \text{sq. inches.}$$

Problem XIII.

Demonstration of the Rule.

By Pr. VI of Pt. III, $C \doteq D \times \Pi = 2R \times \Pi$.

And, by the last Pr., $A \doteq \frac{1}{2}(C \times R)$.

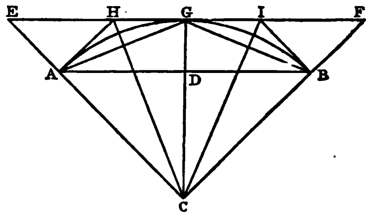
$\therefore A \doteq R^2 \times \Pi$.

Or the rule may be demonstrated *Geometrically*, independently of the two problems just quoted, and the value of Π investigated, by what is called the *Method of Exhaustions*, thus:—

In the first place, we know that the area of a circle must be greater than that of any inscribed and less than that of any circumscribed polygon, and that the greater the number of sides of either, the nearer it will approach the area of a circle. Now, in order to find the area of a polygon of a very great number of sides inscribed in a circle, or of one described about it, the operation will be much facilitated by previously resolving the following problem.

Preliminary Problem. *Having given the area of a regular polygon inscribed in a circle, and of a similar polygon described about it, to find the area of an inscribed and of a circumscribed circle of double the number of sides.*

Let AB be the side of the given inscribed polygon; EF, parallel to AB, that of the similar circumscribed polygon; and C, the centre of the circle. If the chord, AG and the tangents, AH, BI, be drawn, the chord, AG will be the side of an inscribed polygon of double the number of sides; and HI (= 2HG)' the side of a similar circumscribed polygon. Whatever part then the triangles ACD and ECG are of the two given polygons, the same part are the triangle ACG and the quadrilateral ACGHA, of the two required polygons, since the ratio that each of these bears to the whole polygon is the same as that of the angle ACG to four right angles.



Let M be the area of the given inscribed polygon, and N that of the given circumscribed polygon; and let m and n be the areas of the two required polygons.

Then, *first*, the triangles ACD, ACG, being of the same altitude, are to each other as their bases CD and CG (Euc. VI, 1). And they are also to each other as the polygons M and m, of which they are like parts. Therefore $M : m ::$ CD : CG. Again, the triangles CAG, CEG, are to each other as their bases CA and CE, and therefore the polygons m and N (of which these two triangles are like parts) have the same ratio. Therefore $m : N ::$ CA : CE :: CD : CG (Euc. VI, 2, with V, 18) :: $M : m$ (Euc. V, 11). Hence $m^2 \doteq M.N$ (Euc. VI, 16), and $m \doteq \sqrt{(M.N)}$.

Secondly.—The triangles CHG and CHE are to each other as their bases GH and HE. But since, in the Δ ECG, the vertical angle ECG is bisected by CH, we have GH : HE :: GC : CE (Euc. VI, 3) :: CD : CA :: CD : CG :: Δ CAD : Δ CAG :: $M : m$. Therefore Δ CHG : Δ CHE :: $M : m$. Hence, by inversion and composition, Δ CHG + Δ CHE = Δ CGE : Δ CHG :: $M + m : M$; and, doubling the consequents, Δ CGE : 2 Δ CHG = quad. CAHG :: $M + m : 2M$. Now the Δ CGE and the quad. CAHG, are like parts of the polygons N and n. Therefore $N : n :: M + m : 2M$; and, consequently, $n \doteq \dfrac{2M \times N}{M + m}$.

Now, to apply those two results to finding the area of the circle.

The area of a square inscribed in a circle having its radius 1, is 2; and that of a circumscribed square is 4. Therefore we have $M = 2$, and $N = 4$, to commence with; and, by the two formulæ found in the preliminary problem above, $m \doteq \surd(M.N) = \surd(2 \times 4) = \surd 8 = 2{\cdot}8284271$, and $n \doteq \frac{2M \times N}{M + m} = \frac{4 \times 4}{2 + 2{\cdot}8284271} = 3{\cdot}3137085$.

We have thus found the areas of an inscribed and of a circumscribed octagon; and, considering these as now given, we have, for a new operation, $M = 2{\cdot}8284271$, and $N = 3{\cdot}3137085$, and find $m = 3{\cdot}0614674$, and $n = 3{\cdot}1825979$, which are the areas of polygons of sixteen sides. And thus we proceed, finding successively the numbers in the following table.

Number of Sides.	Area of Insc. Polygon.	Area of Circ. Polygon.
4	2·000,000,0	4·000,000,0
8	2·828 427 1	3·313 708 5
16	3·061 467 4	3·182 597 9
32	3·121 445 1	3·151 724 9
64	3·136 548 5	3·144 118 4
128	3·140 331 1	3·142 223 6
256	3·141 277 2	3·141 750 4
512	3·141 513 8	3·141 632 1
1024	3·141 572 9	3·141 602 5
2048	3·141 587 7	3·141 595 1
4096	3·141 591 4	3·141 593 3
8192	3·141 592 3	3·141 592 8
16384	3·141 592 5	3·141 592 7
32768	3·141 592 6	3·141 592 6

Hence it appears that the areas of the inscribed and circumscribed polygons of 32768 sides agree to the first eight figures of the numeral expressions for their value: therefore the numeral expression for the circle itself will agree with the polygons to the extent of the same eight figures, so that, the radius being 1, the area is 3·141,592,6 ±, which expresses the number Π as far as the decimal places are carried.

But the areas of circles being to one another as the squares of their radii, it follows that the area of any circle $\doteq R^2 \times \Pi$.

Exercise 2.

$(379^2 = 143641) \times 3.1416 \doteq 451263$ – sq. links.

Exercise 3.

$$R^2 \doteq 12{\cdot}92^2 = 166{\cdot}93-.$$

Exercise 4.

$$(29{\cdot}5^2 = 870{\cdot}25) \times 3{\cdot}1416 \doteq 2734 - \text{sq. in.}$$

Exercise 5.

$$\left\{\left(\frac{191}{22}\right)^2 = \frac{36481}{484}\right\} \times 3{\cdot}1416 \doteq 236{\cdot}8-.$$

PROBLEM XIV.

Demonstration of the Rule.

By the last problem,

$$\text{Area of circle} \doteq R^2 \times \Pi = D^2 \times \tfrac{1}{4}\Pi.$$

And, by Pr. VI of Ch. III,

$$D \doteq \frac{C}{\Pi}, \text{ or } D^2 \doteq \frac{C^2}{\Pi^2}$$

$$\therefore \text{Area} \doteq \frac{C^2}{\Pi^2} \times \frac{\Pi}{4} = C^2 \times \frac{1}{4\Pi} = \frac{C^2}{4} \div \Pi.$$

The last expression gives us Rule 1; and the preceding, Rule 2, for $1 \div 4\,\Pi \doteq {\cdot}07958$.

Exercise.

By Rule 1, $\frac{1}{4}$ of $(9\frac{1}{5})^2 \doteq \frac{46}{5} \times \frac{46}{5} \times \frac{1}{4} = 21{\cdot}16.$
$21{\cdot}16 \div 3{\cdot}1416 \doteq 6{\cdot}735 +.$

By Rule 2,
$\frac{46}{5} \times \frac{46}{5} \times {\cdot}07958 \doteq 84{\cdot}64 \times {\cdot}07958 = 6{\cdot}736 -.$

The former answer is the more correct, 3·1416 being a closer approximation to the true number than ·07958.

PROBLEM XV.

The *demonstration* of the rule is precisely the same as that of the rule in Problem XII,—the geometrical mode regarding the sector as made up of an infinite number of triangles, having their common altitude equal to the radius, —and the algebraical mode regarding the sector and arc as variable, and the radius as constant, and showing that $\partial s \doteq \frac{1}{2} r \partial z$, and consequently $s = \frac{1}{2} rz$.

NOTE. It is almost unnecessary to say that the result can, in no case, be depended upon to a greater number of figures than warranted by the particular rule selected for finding the arc.

Exercise 2.

Arc, by Pr. IX of Ch. III, ...23·292 – .
Radius, by Pr. VI of Ch. III, 12·414 +.

Exercise 3.

Circumf. = 578 × 3·1416, 1815·84.
Arc, by Pr. IX of Ch. III, 473·17.

Exercise 4.

See the following diagram.

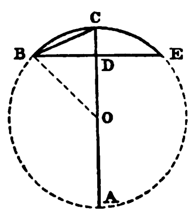
Arc, by Pr. XIII of Ch. III, ...$323\frac{1}{3}$.
$CD = \sqrt{(BC^2 - BD^2)}$, ...85·323.

By Pr. VIII of Ch. III,

Diam. ≑ $BC^2 \div CD$....... = 274·36.
∴ Sector ≑ 68·59 × $323\frac{1}{3}$ = 22 (177 +).

Exercise 5.

$BC^2 = BD^2 + CD^2$
$= 6^2 + 3^2 = 45.$
∴ Diam. $= BC^2 \div CD.$
$\doteq 45 \div 3 = 15.$
$BC \doteq \sqrt{45} = 6{\cdot}708 +.$
Hence, Arc ≑ 13·89 – .



Exercise 6.

BD ÷ BO ≑ 6 ÷ $7\frac{1}{2}$ = ·8000 = N. sine of Arc BC.
Hence, Arc to rad. 1 ≑ ·9273.
Arc BCE ≑ ·9273 × $7\frac{1}{2}$ × 2 = 13·9095.

PROBLEM XVI.

Demonstration of Rule 5.

The number found in the column of segments, in Table II, is the area of a segment of a circle whose radius is 1,

and similar to the segment whose area is required. The areas of the tabular segments have previously been computed by calculating the areas of the sectors for every 20 minutes of the arc, and the areas of the corresponding triangles (represented severally by BOE) for every twenty minutes of the angle BOE, and subtracting the one from the other.

We secure the similarity of the tabular segment to the given segment by taking the same number of degrees, when these are given, or, when the chord is given, by taking a tabular segment having a chord which bears the same ratio to the radius 1 that the given chord bears to the given radius, or, when the height is given, by taking a tabular segment having a height which bears the same ratio to the radius 1 that the given height bears to the given radius. Thus if we express the area of the given segment by a, its radius by r, its chord by c, and its height by h, the area, chord, and height of a similar segment, whose radius is 1, being expressed by a, c, and h. Then, by Pr. XXVI of Ch. III,

$$r : 1 :: c : \mathrm{c} :: \tfrac{1}{2}c :: \tfrac{1}{2}\mathrm{c},$$
$$\text{and } r : 1 :: h : \mathrm{h}.$$
$$\therefore \tfrac{1}{2}\mathrm{c} \doteq \tfrac{1}{2}c \div r, \text{ and } \mathrm{h} \doteq h \div r,$$

which gives us the former part of our rule, or the method of finding the tabular half chord (or sine of half arc) or the tabular height (or versed sine of half arc).

When the tabular sine or versed sine computed does not exactly agree with any one on the table, if we take the nearest on the table we shall obtain a segment differing from the truth by less than the area which is due to a difference of 10 minutes of a degree of the arc. When greater accuracy is required, more explicit directions will be found in the Author's Complete Treatise, Part VII, Prel. Problem.

Having found the area of the corresponding tabular segment, or the similar segment whose radius is 1, we pass from it to the required segment whose radius is r, by the following proportion,

$$1 : r^2 : \mathrm{a} : a,$$

which we obtain by anticipation of Pr. L of this chapter. From this proportion we have $a = \mathrm{a} \times r^2$, which is the concluding part of our rule.

Demonstration of Rule 6.

If we put c for the chord and h for the height of the

segment, d expressing the diameter of the circle, then the area of the segment will be truly expressed by the following series,—

$$4h\sqrt{dh} \times \left\{\frac{1}{3} - \frac{1}{5}\cdot\frac{1}{2}\left(\frac{h}{d}\right) - \frac{1}{7}\cdot\frac{1}{2\cdot4}\left(\frac{h}{d}\right)^2 - \frac{1}{9}\cdot\frac{1.3}{2.4.6}\left(\frac{h}{d}\right)^3 - \&c.\right\}.^*$$

Now since h is always less than d, the fraction $h \div d$ is less than 1, and its ascending powers continually diminish in value as they advance in the series. For a double reason, therefore, the successive terms of the series become less and less, and *that* so rapidly, when h is small compared with d, that the whole value of the series will, in that case, be found to be nearly identical with the sum of two or three leading terms; and in all cases not very far from it. If, therefore, the expression given in Rule 6 can be proved to agree with the above expression as far as two or three terms of the series, it will be shown to give an approximation to the truth (which is all that it professes), and *that* the more close by how much the smaller the segment is.

To show that the two expressions do agree to a certain extent, we have, by Euc. III, 35 (referring to the diagram in the *Course*), $BD^2 = CD \times DA$, or

$$\tfrac{1}{4}c^2 = h(d-h) = hd - h^2.$$

$$\therefore \tfrac{1}{4}c^2 + \tfrac{4}{10}h^2 \doteq hd - h^2 + \tfrac{4}{10}h^2 = hd - \tfrac{3}{5}h^2.$$

$$\therefore \tfrac{4}{3}h \times \sqrt{(\tfrac{1}{4}c^2 + \tfrac{4}{10}h^2)} \doteq \tfrac{4}{3}h \times \sqrt{(hd - \tfrac{3}{5}h^2)}$$

$$= 4h\sqrt{hd} \times \frac{1}{3}\sqrt{\left(1 - \frac{3}{5}\cdot\frac{h}{d}\right)}$$

$$= 4h\sqrt{hd} \times \left\{\frac{1}{3} - \frac{1}{5}\cdot\frac{1}{2}\left(\frac{h}{d}\right) - \frac{3}{25}\cdot\frac{1}{2.4}\left(\frac{h}{d}\right)^2 - \&c.\right\},$$

as is found by extracting the latter root in the previous member of the equation.

Now the series just found agrees, it will be observed, with the true series given above, in two terms, and differs in the third term only by having $\frac{3}{25}$ instead of $\frac{1}{7}$ or $\frac{3}{21}$, giving an error of $+\frac{1}{350}\left(\frac{h}{d}\right)^2$ in that term, a fraction which, as well as all the subsequent terms, is trifling compared with the values of the previous terms when h is small compared with d.

* The demonstration of this may be found in the Key to the Author's Complete Treatise (Part IV, Pr. XVI, Rule III), and in various other works.

The total error, in employing Rule 6, will not be more than 1 in 23000 if the segment does not exceed 60°; or 1 in 4560, if not exceeding 90°. But it will amount to $1-1360^{th}$ part of the whole if the segment is of 120°, and will exceed $1-231^{st}$ part if the segment is greater than a semicircle. This rule gives a result always greater than the truth.

The fraction $\frac{4}{10}$ is not reduced to its lowest terms, because $\frac{4}{10}$ is an easier multiplier than $\frac{2}{5}$. For a similar reason the factor 2, external to the root, is not permitted to pair off with $\frac{1}{4}$ internal to the radical sign, since $\frac{1}{4}c^2$ is often as easily found as c^2.

Exercise 1.

By Rule 5. Looking for 90° in the right-hand column of Table II, we find, in the same line, in the column of segments, ·2854. This we multiply by 24^2 or 576.

Exercise 2.

Tabular Seg. of 60° ≑ ·0906. This, multiplied by Radius ^{2} or ·02533, gives ·002295 –.

Exercise 3.

Tabular N. sine ≑ $12 \div 12\frac{1}{2}$...... = ·9600.
Nearest on Table, ... ·9596.
Corresponding tab. Segment, ... 1·0159.
Required Seg. ≑ $1·0159 \times (12\frac{1}{2})^2 = 158·7 +$.

Exercise 4.

Tabular N. V. S. ≑ $27 \div 37·5$ ..... = ·7200.
Nearest on Table,.................... ·7188.
Corresp. tabular Segment,......... 1·0159.
Req. Segment ≑ $1·0159 \times 1406·25 = 14(29 -)$.

Exercise 5.

(See the diagram in the *Course.*)
AD ≑ $10^2 \div 4 = 25$. Hence
Diam. ≑ $25 + 4 = 29$, and rad. = $14\frac{1}{2}$.
N. Sine ≑ $10 \div 14\frac{1}{2}$ .... = ·6897
Nearest on Table, ... ·6905
Corresponding tabular Seg...... ·2627
Required Seg. ≑ $·2627 \times 210·25 = 52·(1 -)$.

Exercise 6.

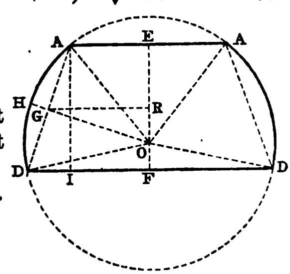
$c \doteq 20$, and $h \doteq 4$.
$\frac{1}{4}c^2 \doteq 100$
$\frac{4}{10}h^2 \doteq 6{\cdot}4$
$\sqrt{106{\cdot}4} \doteq 10{\cdot}3150.$
$10{\cdot}315 \times 4 \times \frac{4}{3} \doteq 55{\cdot}0135.$

Problem XVII.

No *demonstration* is required.

Exercise 1.

$AD \doteq \sqrt{(AI^2 + DI^2)} = \sqrt{(21^2 + 3^2)} = \sqrt{450} = 21{\cdot}2132.$
$DG \doteq \frac{1}{2} AD = 10{\cdot}6066.$
$DO \doteq \sqrt{(DF^2 + OF^2)}$
$= \sqrt{(144 + 81)} = 15.$



Then finding the segment AHD, by Rule 5 of the last Problem,
N. Sine $\doteq$ DG $\div$ DO = ·70705.
Nearest on Table, ·7071.
Cor. tab. Seg........ ·2854

$$\left.\begin{array}{r}{\cdot}2854 \times 225 \doteq 64{\cdot}215 \\ 64{\cdot}215\end{array}\right\} = \text{Segments.}$$
$$441{\cdot}000 = \text{Trapezoid.}$$
$$\underline{569{\cdot}43} = \text{Zone.}$$

Exercise 2.

GH $\doteq 10 - 7\frac{1}{2} = 2\frac{1}{2}$, since, in this case, OG is parallel to AA or DD, and equal to the half of either.

$OF \doteq \sqrt{(10^2 - 7{\cdot}5^2)}$ ..... = 6·6144.
$EF \doteq 2\,OF$ .... = 13·2288.
Nat. V. Sine $\doteq 2{\cdot}5 \div 10$ = ·2500.
Hence tabular Segment $\doteq$ ·2267.

$$\begin{aligned} \text{Segments} &= 22{\cdot}67 \times 2 \doteq 45{\cdot}34 \\ \text{Trapezoid} \ldots\ldots\ldots\ldots\ldots &\doteq 198{\cdot}43 \\ \text{Zone} \ldots\ldots\ldots\ldots\ldots\ldots\ldots &\doteq 243{\cdot}77 \text{ sq. in.} \end{aligned}$$

Exercise 3.

$$\text{Area of Circle} \doteq 150^2 \times \tfrac{1}{4}\Pi = 17671{\cdot}5.$$

But AOA is an equilateral triangle: therefore the ∠ AOA is of 60°, and so also is each of the angles AOD and AOD, DOD being, in this instance, a straight line. Therefore the two sectors AOD, AOD, are each one-sixth part of the circle, and the two together are one-third of the circle.

$$\begin{aligned} \therefore \text{Sectors AOD, AOD} &\doteq 5890{\cdot}5. \\ \text{Triangle AOA} \ldots\ldots\ldots &\doteq 2435{\cdot}7. \\ \text{Zone*} \ldots\ldots\ldots\ldots\ldots\ldots &\doteq 8326{\cdot} +. \end{aligned}$$

PROBLEM XVIII.

No *demonstration* of the rule is necessary. The formulæ are obtained from the rule thus:—

Calling the area of the outer circle A, and that of the inner circle a, the area of any ring will be $A - a$.

Now, by Pr. XIII, $A \doteq R^2 \times \Pi$, and $a \doteq r^2 \times \Pi$.

$$\therefore A - a \doteq (R^2 - r^2) \times \Pi.$$

Again, by Pr. XIV, $A \doteq C^2 \times \dfrac{1}{4\Pi}$, and $a \doteq c^2 \times \dfrac{1}{4\Pi}$.

$$\therefore A - a \doteq (C^2 - c^2) \times \frac{1}{4\Pi}.$$

Lastly, since $C^2 - c^2 \doteq (C + c) \times (C - c)$
$= (C + c) \times 2R\Pi - 2r\Pi)$,

$$A - a \doteq (C + c) \times (R - r) 2\Pi \times \frac{1}{4\Pi} = \tfrac{1}{2}(C + c) \times (R - r).$$

Exercise 1.

By the first Formula,

$$\text{Area} \doteq (5^2 - 4^2) \times \Pi = 9 \times \Pi = 28{\cdot}274 +.$$

* In this solution a slight deviation has been made from the rule. It was evidently needless to subtract the triangles from the sectors, in order to find *the segments,* since the same triangles were again to be added as parts of the *trapezoid.*

Exercise 2.

By the second Formula,

$$\text{Area} \doteq (400^2 - 357^2) \times \cdot 07958$$
$$= 32551 \times \cdot 07958 = 2590 +.$$

PROBLEM XIX.

The rule again requires no *demonstration*, being self-evident.

Exercise.

The easiest mode of computing the segments is by Rule 6 of Problem XVI.

Outer Segment.	Inner Segment.
$\frac{1}{4}c^2 \doteq 24^2 = 576$	$\frac{1}{4}c^2 \doteq 576$
$\frac{4}{10}h^2 \ldots\ldots \doteq 40$	$\frac{4}{10}h^2 = 19 \cdot 6$
$24 \cdot 8193 \doteq \sqrt{616}.$	$24 \cdot 4049 \doteq \sqrt{595 \cdot 6}.$
$24 \cdot 8193 \times \frac{4}{3} \text{ of } 10 \doteq 330 \cdot 9$	$24 \cdot 4049 \times \frac{4}{3} \text{ of } 7 \doteq 227 \cdot 8.$
$227 \cdot 8$	
Ans. $103 \cdot 1.$	

PROBLEM XX.

Demonstration of the Rule.

The convex surface of a right cylinder or prism is really nothing else than a rectangle. This will be perceived at once by bending a rectangular piece of paper or card-board into a cylinder or prism, or by covering a right cylinder or prism with paper and then unwrapping it. Of that rectangle the length is L and the breadth C: therefore its area is $C \times L$.

If more vigorous demonstration be called for,—Since the sides of a right *rectilineal prism* are all rectangles having the same length L, but different breadths; call their breadths D, E, F, &c. Then their areas will be $D \times L$, $E \times L$, $F \times L$, &c.; and their united area will be $(D + E + F + \&c.) \times L = C \times L$. In the case of the right cylinder

or curvilineal prism, the number of rectangles is infinite, but the demonstration the same.

Exercise 2.

Circumf. $\doteq 32$ in. $\times \Pi$......	$=$	$100{\cdot}531$
Convex surface.............	$=$	$9047{\cdot}79$
Area of one end $\doteq 16^2 \times \Pi$	$=$	$804{\cdot}25$
Area of the other end......	$\doteq$	$804{\cdot}25$
		$10656{\cdot}29$ sq. in.

Exercise 3.

Each end $\doteq 2{\cdot}0449 \times 2{\cdot}5981 = 5{\cdot}3128$.
Six sides $\doteq 6 \times 8{\cdot}5 \times 1{\cdot}43 = 51 \times 1{\cdot}43 = 72{\cdot}93$.

Problem XXII.

Demonstration of the Rule.

The reason of this rule, *in the case of the cone*, may be explained simply in the same manner as that of the rule for the right-cylinder. Let the convex surface of a right cone be covered with paper, carefully cutting off every part that overlaps, or reaches beyond the convex surface. On spreading the paper flat again, it will be found that it forms exactly a sector of a circle, the circumference of the cone's base becoming the arc of the sector, and the slant height of the cone becoming the radius. The rule for its area will therefore be the same as that for the sector.

In the case of the regular pyramid, the area is the sum of all the triangular sides, which, having one altitude, have the sum of their areas equal to half the sum of their bases multiplied by the common altitude,—the altitude of each triangle being the slant height of the pyramid. This, if we choose, may be expressed in more precise algebraical language in the same manner as in the demonstration of the last problem. The same demonstration may also be extended to the cone, by regarding it as a pyramid of an infinite number of sides.

Exercise 2.

$54\tfrac{5}{12} \times 50 \doteq \tfrac{653}{12} \times \tfrac{50}{1} = 2721 -$ sq. feet.
160

Exercise 3.

Half circumf. $\doteq 2{\cdot}9 \times \pi = 9{\cdot}1106 +$.
$9{\cdot}1106 \times 53{\cdot}7 \doteq 66{\cdot}96 +$.

Exercise 4.

Slant height $\doteq \sqrt{(20^2 + 15^2)} = \sqrt{625} = 25$.

Conv. surf. $\doteq 15\,\pi \times 25 = 375\,\pi = 1178{\cdot}1$
Base $\doteq 30^2 \times {\cdot}7854$ .... $= 706{\cdot}9$
Whole surface .... $\doteq 1885{\cdot}0$.

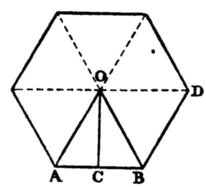
Exercise 5.

Let the hexagon in the annexed diagram represent the base of the pyramid. In this

$$OC^2 \doteq OA^2 - AC^2 = 100 - 25 = 75.$$

We may now suppose the pyramid to be constructed on this base, having its vertex V,* and its slant height VC. Then in $\triangle$ VOC,

$$\begin{aligned} VC &\doteq \sqrt{(VO^2 + OC^2)} \\ &= \sqrt{(400 + 75)} \\ &= \sqrt{(475} = 21{\cdot}7945 - . \end{aligned}$$



Surf. of 6 sides $\doteq 21{\cdot}7945 \times 30 = 653{\cdot}8350$
Base $\doteq 100 \times 2{\cdot}5981$ .... $= 259{\cdot}81 -$

Answer, $913{\cdot}64 +$.

PROBLEM XXIII.

Demonstration of the Rule.

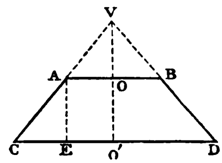
Let S be the slant height of the whole pyramid; s that of the part cut off; and s' that of the remaining frustum. Then, by the last problem,

* The point V is not represented in the diagram, but may be readily conceived as raised vertically above O.

Convex surface of whole pyramid .... $\doteq \frac{1}{2}C.S.$
C. surf. of part cut off $\doteq \frac{1}{2}cs = \frac{1}{2}c(S-s') = \frac{1}{2}cS - \frac{1}{2}cs'$.
$\therefore$ C. surf. of frust. $\doteq \frac{1}{2}C.S - \frac{1}{2}cS + \frac{1}{2}cs' = \frac{1}{2}(C-c)S + \frac{1}{2}cs'$.
But, from similar triangles,

$$C : c :: S : s$$
$$\therefore C : C-c :: S : S-s = s'$$
$$\therefore (C-c)S \doteq Cs'.$$

$\therefore$ Conv. surf. of frustum
$$\doteq \tfrac{1}{2}Cs' + \tfrac{1}{2}cs'$$
$$= \tfrac{1}{2}(C+c)s'.$$



Exercise 1.

$$3\tfrac{5}{8} \times 2\tfrac{3}{4} \doteq 6\tfrac{3}{8}.$$
$$6\tfrac{3}{8} \times 5\tfrac{3}{8} \div 2 \doteq \tfrac{51}{8} \times \tfrac{43}{8} \times \tfrac{1}{2} = 17{\cdot}13 +.$$

Exercise 2.

Circumf. of base $\doteq 26{\cdot}075$
.... of top $\doteq 18{\cdot}850$

$\frac{1}{2}$ of $7{\cdot}4 \times 44{\cdot}925 \doteq 166{\cdot}22 +$.

Or, more easily thus:—
By the first formula for the frustum of a cone,

$$\text{Area} \doteq \tfrac{1}{2}(8{\cdot}3 + 6) \times 7{\cdot}4 \times \Pi = 14{\cdot}3 \times 3{\cdot}7 \times \Pi.$$
$$= 52{\cdot}91\,\Pi = 166{\cdot}22 +.$$

Exercise 3.

Diam. of base $\doteq 3 \times {\cdot}31831 = {\cdot}95493$
Diam. of top $\doteq 2 \times {\cdot}31831 = {\cdot}63662$
Difference $= 1 \times {\cdot}31831 = {\cdot}31831$*.

$$\text{CE} = \tfrac{1}{2} \text{ of } {\cdot}31831 \doteq {\cdot}159155.$$

$\text{CE}^2 \doteq 0{\cdot}02533$	AC $\doteq 1{\cdot}0126-$
$\text{AE}^2 \doteq 1{\cdot}0$	$C+c \doteq$ 5
$\text{AC}^2 \doteq 1{\cdot}02533$	2)5·0630 −

Convex surface .... $\doteq 2{\cdot}5315 -$.
Area of base $\doteq 9 \times {\cdot}07958 = {\cdot}7162 +$
Area of top $\doteq 4 \times {\cdot}07958 = {\cdot}3183 +$
Total, $3{\cdot}5660 \pm$.

* It is obvious that the two products above this need not have been found. They are put down for the sake of younger pupils to whom the more concise *process may not be familiar.*

Exercise 4.

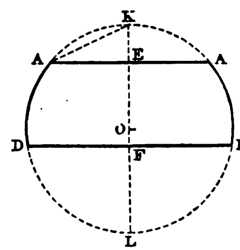
$$\text{Area} \doteq (15 + 5) \times 6 \times 74 \div 2 = 60 \times 74.$$

Problem XXVI.

Demonstration of the Rules.

The rules for this and the subsequent problem may be demonstrated thus :—

Let R be the radius of the sphere; D, its diameter; C its circumf.; s, the surface of a segment AKA, and z the arc AK. Then AE $\doteq$ sin z, and KE = versin z: and, by a principle familiarly employed in the application of the Integral Calculus to the quadrature of surfaces bounded by curves,



$$\partial s \doteq 2\pi \sin z \, \partial z$$
$$= 2\pi R \, \partial \text{ versin } z,$$

Since $R \, \partial$ versin $z \doteq \sin z \, \partial z$.

$$\therefore s \doteq 2\pi R \text{ versin } z,$$

the constant being = 0. Therefore, since $2\pi R \doteq C$,

Surface of segment AKA $\doteq C \times$ KE, and

Surface of segment DKD $\doteq C \times$ KF.

$\therefore$ Surf. of Zone AADD $\doteq$ C $\times$ (KF $-$ KE) = $C \times$ EF.

Again, let z become the semicircumf. Then versin z becomes the diameter, and s becomes the surface of the whole sphere.

$$\therefore \text{Surface of sphere} \doteq \text{C} \times \text{D} = \text{D}^2 \times \pi.$$

Cor. 1. Hence the surface of any zone of a sphere is equal to that of a segment of equal height.

Cor. 2. The surfaces of any two zones of a sphere, of equal heights, are equal.

Cor. 3. The surfaces of segments and zones of the same sphere are to one another as their heights.

Cor. 4. The surface of a sphere is equal to four times the area of a great circle, and the convex surface of a hemisphere is double the area of its base.

Cor. 5. The surface of any segment (AKA) of a sphere is equal to the area of a circle whose radius is the chord

(AK) of half the arc of the segment. For $C \times KE \doteq D \times \Pi \times KE = AK^2 \times \Pi$.

Cor. 6. The surface of a sphere is equal to the convex surface of the circumscribed cylinder; and planes, perp. to the axis of the circumscribed cylinder, cut off or intercept equal surfaces on the sphere and cylinder.

Exercise 3.

By the second rev. formula of Pr. VI, Ch. III.

$$\text{Diam.} \doteq 1 \times \cdot 31831.$$

Problem XXVII.

The *demonstration* is contained in that of the preceding problem.

Exercise 1.

$$\text{Circumf. of sphere} \doteq 823\cdot 8 \times \Pi = 2588\cdot 044.$$

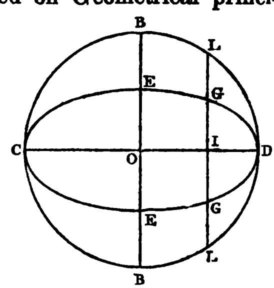
This and the final result will come out a little different if we limit Π to five figures.

Exercise 2.

$$\text{Height of zone} \doteq 59 - (12 + 14) = 33 \text{ in.}$$
$$\text{Circumf. of sphere} \doteq 59\,\Pi = 185\cdot 354 \text{ in.}$$

Problem XXX.

The rule may be demonstrated on Geometrical principles, thus:—



Let a circle be described on one of the axes (CD) of the ellipse, as a diameter; and let a st. line LL move from D to C, keeping parallel to EE, the other axis. In every part of its progress the circular double ordinate, LL, will be to the elliptic double ordinate, GG, as BB to EE. Therefore the *circle will* have that same ratio to the ellipse, both being

generated by the motion of the corresponding double ordinates in the same time; and any segment (LDL) of the circle, cut off by the double ordinate, will have the same ratio to the corresponding segment (GDG) of the ellipse.

If the idea of motion be objected to, suppose the circle to consist of an infinite number of infinitely narrow rectangles parallel to BB, each of which may be represented by the line LL: the ellipse will be made up of an infinite number of rectangles, each of which may be represented by GG. But every one of these rectangles in the circle will be to the corresponding rectangle in the ellipse in the uniform ratio of LL to GG. Therefore the whole circular space, made up of the former class of rectangles, will bear the same ratio to the elliptic space, made up of the same number of the latter class of rectangles, that CD bears to EE.

Now, allowing $2a$ to represent the axis on which the circle is described, viz. CD, and $2b$ the other axis, EE. Then

$$\text{Circle} : \text{Ellipse} :: 2a : 2b.$$

$$\therefore \text{Ellipse} \doteq \frac{b}{a} \times \text{Circle} = \frac{b}{a} \times a^2 \Pi = ab\,\Pi.$$

Or, by the higher Calculus, thus:—

Let s represent the elliptic segment, GDG; s, the corresponding circular segment, LDL; and v the variable abscissa, ID. Then by a principle well known in the application of the Integral Calculus to the quadrature of surfaces bounded by curve lines.

$$\partial \mathrm{s} \doteq \mathrm{LL} \times \partial v, \text{ and } \partial s \doteq \mathrm{GG} \times \partial v.$$

$$\therefore \partial \mathrm{s} : \partial s :: \mathrm{LL} : \mathrm{GG}.$$

But, by a familiar property of the ellipse,

$$\mathrm{LL} : \mathrm{GG} :: \mathrm{BB} : \mathrm{EE} :: 2a : 2b :: a : b.$$

$$\therefore a : b :: \partial \mathrm{s} : \partial s.$$

$$\therefore \partial s \doteq \frac{b}{a} \times \partial \mathrm{s}.$$

Integrating both members of the equation, and keeping in view that there is no constant since both segments vanish together, when v becomes 0, we have

$$s = \frac{b}{a} \times \mathrm{s}.$$

When v becomes = CD, then s becomes the whole circle, and s the whole ellipse.

$$\therefore \text{Ellipse} \doteq \frac{b}{a} \times \text{circle} = \frac{b}{a} \times a^2 \Pi = ab\,\Pi = 2a \times 2b \times \frac{\Pi}{4}.$$

Exercise 2.

$$\text{Area} \doteq 1{\cdot}25 \times {\cdot}74 \times \Pi = {\cdot}925 \times \Pi = 2{\cdot}906 - \text{sq. ch.}$$

Problem L.

Demonstration of the Rule.

The rule, in the case of similar triangles with reference to their corresponding sides, is proved in Euclid, B. VI, Pr. XIX, and, in the case of any similar rectilineal figures in reference to their corresponding sides, in Pr. XX, or at least by an easy inference from these propositions. And, if the corresponding lines are not sides, the proof is completed with regard to these, by connecting them with corresponding sides by one or more triangles, by means of which the said lines will be shown to have the same ratio to each other as the corresponding sides of the figures.

In the case of circles and their diameters, the proof is given in Euclid, B. XII, Pr. II; and their radii are as their diameters. From this case we easily pass to that of similar sectors of circles, since these are like parts of the whole circles: and if from these we deduct the triangles which have likewise the same ratio, the rule is proved for similar segments with reference to the diameters or radii of their circles. For similar segments with reference to their heights, bases, or other lines, it has been shown, in Pr. XXVI of Ch. IV, that these have to each other the same ratio as the diameters or radii. Hence we readily advance to any other spaces bounded by circular arcs, or by straight lines and circular arcs.

The surfaces of cubes, parallelopipeds, and other prisms are made up of rectangles and triangles. So also are those of wedges, pyramids, and all other solids bounded by planes. The surfaces of cylinders, cones, and frustums of cones, are rectangles, sectors of circles, and similar portions of sectors. And the proof in the case of all these has just been given.

In the cases of the surfaces of spheres, spherical segments and zones, ungulae of cones and of cylinders, ellipses, parabolas, hyperbolas, and their zones and segments, spheroids, paraboloids, hyperboloids, and their zones and segments, the truth of the rule may be proved from the

particular rules given in the previous problems of this chapter, and in the Author's *Complete Treatise.*

In the case of plane figures bounded by other curves, regular or irregular, we regard minute portions of these as straight lines, and in the case of solid figures bounded by curve surfaces different from those mentioned, regular or irregular, we suppose these surfaces to be made up of minute plane surfaces. The *universality* of the rule follows from its being true as applied to all similar *rectilineal* figures, however varied or however minute their parts.

Exercise 1.

$125^2 : 80^2 :: 2$ ac. $23\frac{3}{4}$ pls. : 3 ro $20\frac{4}{5}$ pls.

Exercise 2.

$$100^2 : (17\tfrac{4}{11})^2 :: 31416 : \text{——}.$$
$$(17\tfrac{4}{11})^2 \doteq (\tfrac{191}{11})^2 = \tfrac{36481}{121}.$$
$$\tfrac{36481}{121} \times 31416 \doteq \tfrac{36481}{11} \times 2856 = 947{\cdot}2.$$

Exercise 3.

$50^2 : 40^2 :: 0{\cdot}25$ ac. : $0{\cdot}16$ ac. $= 25{\cdot}6$ pls.

Exercise 4.

$$10^2 : 72^2 :: £5\tfrac{1}{2} : £285{\cdot}12.$$
$$285{\cdot}12 - 5{\cdot}5 \doteq 279{\cdot}62 = £279 : 12 : 5\text{—}.$$

CHAPTER V.

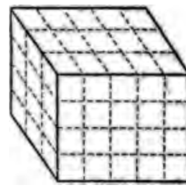
MENSURATION OF SOLIDS.

PROBLEM I.

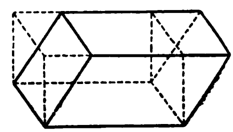
Demonstration of the Rule.

The reason of the rule may be demonstrated to the pupil, in the case of a rectangular parallelopiped, by showing him, with a model, how it may be divided into as many laminæ, or boards, as there are units in its height,

and how each lamina or board may be divided into as many rectangular rods, or bars, as there are units in its breadth, and each rod or bar into as many cubes, or cubic units, as there are units in its length. Supposing the length of the solid to be 5, its breadth 3, and its height 4, we shall then have 4 boards, each board containing 3 rods, and each rod 5 cubes, or cubic units: the whole number of cubes in a rod will therefore be 5; in a board, 3 times 5, or 15, which is the area of the base; and in the whole, 4 times 15.



A parallelopiped may then be taken with a rectangular base, and the sides perpendicular but the ends oblique to the base, which, it may be shown, can be converted into a rectangular parallelopiped by cutting a wedge from the one end and putting it to the other. The conversion is equally easy if the ends are perpendicular to the base and the sides oblique.



The same thing may next be shown with a parallelopiped whose base is still rectangular, but the sides and ends *both* oblique to the base: for, by cutting a wedge from one side and putting it to the other, it will be changed into a solid similar to that described in the previous case.

Lastly, if the parallelopiped has its six sides all oblique-angled parallelograms, by cutting a wedge from the one end and putting it to the other, it is converted into a parallelopiped with a rectangular base as in the last case.

In all these cases it will appear that the oblique parallelopiped is equal to a rectangular one of the same length, and of the same breadth and thickness measured perpendicularly to the length.

In the case of cylinders and other prisms, right and oblique, it may be pointed out how these, if made of soft material, might be moulded into rectangular parallelopipeds, of equal altitudes, without changing the area of the base, or of any section parallel to the base; and that, consequently their content may be found by the same rule.

For the strict geometrical demonstration of the equality of the oblique to the rectangular parallelopipeds the student is referred to Euclid's Elements, B. XI, Pr. XXXI. *Triangular prisms* and their bases are the halves of parallelo-

pipeds and their bases; and all other rectilineal prisms are made up of triangular prisms. Cylinders, and other prisms not rectilineal, may be supposed to consist of an infinite number of triangular prisms. Or, rectilineal prisms may be inscribed in them, or described about them, so close to their outlines as to differ from them as little as we please; and therefore since the rule holds good as applied to such rectilineal prisms, it must also hold good in the case of the cylinders and other prisms.

A general demonstration for all cases might be given very simply by the higher Calculus; but it would necessarily assume a principle similar to that employed in the method just described.

Exercise 1.

$$1{\cdot}25^3 \doteq 1{\cdot}953125 \text{ c. foot.}$$
$$\cdot953125 \times 1728 \doteq 1647 \text{ c. in.}$$

Or thus:—

$$(1\tfrac{1}{4})^3 \doteq (\tfrac{5}{4})^3 = \tfrac{125}{64} = 1\tfrac{61}{64} = 1\tfrac{1647}{1728}.$$

Exercise 2.

$$\begin{array}{rrrr} \text{Ft.} & \text{in.} & & \\ 3 : & 4 & & \\ \times 3 : & 4 & & \\ \hline 10 : & 0 & & \\ 1 : & 1 : & 4 & \\ \hline 11 : & 1 : & 4 & \\ \times 3 : & 4 & & \\ \hline 33 : & 4 : & 0 & \\ 3 : & 8 : & 5 : & 4 \\ \hline 37 : & 0 : & 5 : & 4 \\ \hline\hline \end{array}$$

Or thus:—

$$(3\tfrac{1}{3})^3 \doteq (\tfrac{10}{3})^3 = \tfrac{1000}{27}.$$

$$\begin{array}{l} \quad\text{Ft.} \\ 3)1000 \\ \hline 9)333 : 4 \\ \hline \quad 37 : 0 : 5 : 4. \\ \hline\hline \end{array}$$

Exercise 3.

$$66 \times 12 \times 12 \doteq 9504 \text{ c. feet.}$$
$$9504 \times 180 \doteq 1710720 \text{ lb} \doteq 15274 + \text{cwt.}$$

Or, more concisely thus:—

$$\frac{66 \times 12 \times 12 \times 180}{28 \times 4 \times 20} \doteq \frac{66 \times 3 \times 3 \times 9}{7} = 764 - \text{tons.}$$

Exercise 6.

$$\text{Area of base} \doteq 9 \times 2{\cdot}5981 = 23{\cdot}383 - .$$

Exercise 7.

$$\text{Area of base} \doteq (1\tfrac{1}{4})^2 \times \frac{\Pi}{4} = \frac{25}{16} \times \frac{\Pi}{4} = 1{\cdot}2272 - .$$

Exercise 8.

$$A \doteq \sqrt{(4{\cdot}5 \times 2{\cdot}5 \times 1{\cdot}5 \times 0{\cdot}5)} = \sqrt{(8{\cdot}4375)} = 2{\cdot}90474 - .$$

Exercise 9.

By the reverse formula for the cube,

$$L \doteq \sqrt[3]{15{\cdot}625} = 2{\cdot}5.$$

Exercise 10.

Content of smaller cube $\doteq 27$.
Content of larger cube $\doteq 54$.
$\sqrt[3]{54} \doteq 3{\cdot}78 -$.

PROBLEM II.

The demonstration of this rule, in the case of a triangular pyramid, may be found in Euclid, B. XII, Pr. VII. In *that* it is proved that a triangular pyramid is the third part of a prism of the same base and altitude. The same thing may be shown mechanically by actually dividing a triangular prism as directed in the proposition just quoted.

And since every other rectilineal pyramid is made up of triangular pyramids, the content of every one of which will be a third part of that of the corresponding prism, the content of the whole pyramid will be one-third of the content of a prism of the same base and altitude. The rule may then be applied to pyramids not rectilineal by supposing them to be made up of an infinite number of rectilineal pyramids, or to be themselves rectilineal pyramids with an infinite number of sides.

A geometrical demonstration, in the case of the cone is given in Euclid, B. XII, Pr. X.

The general demonstration by the higher Calculus is this:—

Let s be the solidity of a variable segment of the cone or pyramid, cut off by a plane parallel to the base; and let x be the height of that segment, and y its base; H

being the height of the whole cone or pyramid, and A its base. Then

$$H^2 : x^2 :: A : y.$$

$$\therefore y \doteq \frac{x^2}{H^2} A, \text{ and } \partial s = y\partial x \doteq A \frac{x^2}{H^2} \partial x.$$

$$\therefore s \doteq \tfrac{1}{3} A \frac{x^3}{H^2} + (C=0).$$

Let x become H, $\because s$ becomes S.

$$\therefore S \doteq \tfrac{1}{3} A \times H.$$

Exercise 1.

$$S \doteq \frac{3 \times 3 \times 8}{3} \times \Pi = 24\,\Pi = 75\cdot40 -.$$

Exercise 2.

$$S \doteq \frac{460 \times 460 \times 315}{3 \times 27} = \frac{460 \times 460 \times 35}{9} = \frac{7406000}{9}.$$

Exercise 3.

$$A \doteq \tfrac{5}{2} \times \tfrac{5}{2} \times 2\cdot5981 = 16\cdot2381.$$
$$S \doteq 16\cdot2381 \times 5 \div 3 = 27\cdot0635 \text{ c. ft.}$$
$$27\cdot0635 \times 170 \doteq 4601 - \text{lb.}$$

Exercise 4.

Diameter of base $\doteq 10\cdot5$. Radius $\doteq 5\cdot25$.

$$\therefore H \doteq \sqrt{(8\cdot75^2 - 5\cdot25^2)} = 7.$$
$$A \doteq 10\cdot5^2 \times \tfrac{1}{4}\Pi = 86\cdot590.$$

PROBLEM III.

Demonstration of the Rule.

Let H be the height of the whole cone or pyramid; h the height of the part cut off; and h' that of the frustum. Then

$$E : e :: H^2 : h^2 \text{ (Ch. IV, Pr. I).}$$
$$\therefore \sqrt{E} : \sqrt{e} :: H : h \text{ (Euc. VI, 22).}$$
$$\therefore \sqrt{E} - \sqrt{e} : \sqrt{E} :: H - h = h' : H.$$
$$\text{and } \sqrt{E} - \sqrt{e} : \sqrt{e} :: H - h = h' : h.$$
$$\therefore H \doteq \frac{\sqrt{E}}{\sqrt{E} - \sqrt{e}} \times h', \text{ and } h \doteq \frac{\sqrt{e}}{\sqrt{E} - \sqrt{e}} \times h'.$$

Now, by the last problem,

$$\text{Solidity of whole cone*} \doteq \frac{E \times H}{3} = \frac{E\sqrt{E}}{\sqrt{E}-\sqrt{e}} \times \frac{h'}{3}.$$

$$\text{Solidity of cone* cut off} \doteq \frac{e \times h}{3} = \frac{e\sqrt{e}}{\sqrt{E}-\sqrt{e}} \times \frac{h'}{3}.$$

$$\therefore \text{Frust.} \doteq \frac{E\sqrt{E}-e\sqrt{e}}{\sqrt{E}-\sqrt{e}} \times \frac{h'}{3} = (E+e+\sqrt{Ee}) \times \frac{h'}{3}.$$

Exercise 1.

$$4+3+\sqrt{12} \doteq 10{\cdot}464+.$$
$$10{\cdot}464 \times 1\tfrac{3}{4} \div 3 \doteq 6{\cdot}104.$$

Exercise 2.

$$\tfrac{49}{4} + \tfrac{25}{4} + \sqrt{(\tfrac{49}{4} \times \tfrac{25}{4})} \doteq 12\tfrac{1}{4} + 6\tfrac{1}{4} + 8\tfrac{3}{4} = 27\tfrac{1}{4}.$$
$$27\tfrac{1}{4} \times 3 \div 3 \doteq 27\tfrac{1}{4}.$$

Exercise 3.

$$\begin{array}{lr}
745^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \doteq & 555025 \\
32^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \doteq & 1024 \\
\sqrt{(745^2 \times 32^2)} = 745 \times 32 \doteq & \underline{23840} \\
 & 579889 \\
\tfrac{1}{3}h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \doteq & 160 \\
27 \begin{cases} 9)\overline{92782240} \text{ c. ft.} \\ 3)\overline{10309138}- \end{cases} & \\
 & 3436379+\text{c. y.}
\end{array}$$

$$\frac{92782240 \times 150}{28 \times 4 \times 20} \doteq \frac{2899445 \times 15}{7} = 6213096 \text{ tons.}$$

Exercise 4.

$$E \doteq 201{\cdot}06. \qquad Ee \doteq 22739+.$$
$$e \doteq 113{\cdot}10. \qquad \sqrt{Ee} = 150{\cdot}80-.$$

Problem VII.

Demonstration of the Rule.

If the lengths of the edge and of the base are equal, the wedge is half of a parallelopiped whose base and height are the same as those of the wedge. The solid content

* Or pyramid.

of such a wedge is therefore $\frac{1}{2}L.\,B.\,H$, or $\frac{1}{2}L'.\,B.\,H$, or $\frac{1}{6}(2L+L').\,B.\,H$.

But if L is greater than L', the wedge evidently exceeds the half parallelopiped, whose length is L' and content $\frac{1}{2}$ $L'.\,B.\,H$, by a pyramid whose base is $L-L'$ in length, whose height, and breadth of base, are the same as those of the wedge, and whose solidity is $\frac{1}{3}$ $(L-L').\,B.\,H$. Therefore, in this case, the content of the wedge is $\frac{1}{3}(L-L').B.H+\frac{1}{2}L'.B.H=(\frac{1}{3}L+\frac{1}{6}L').B.H=\frac{1}{6}(2L+L').\,B.\,H$.

Lastly, when L is less than L', the wedge is less than the half parallelopiped whose length is L' and content $\frac{1}{2}L'.\,B.\,H$, by a pyramid whose content is $\frac{1}{3}(L'-L).\,B.\,H$. Therefore, in this case, the content of the wedge is $\frac{1}{2}L'.\,B.\,H-\frac{1}{3}(L'-L).\,B.\,H=(\frac{1}{3}L+\frac{1}{6}L').\,B.\,H=\frac{1}{6}(2L+L').\,B.\,H$.

Exercise 1.

2 ft. 8 in.	1 ft. 9 in. × 1 ft. 3 in. ≐ 2 ft. 4 pts.
2 „ 8 „	9 ft. 10 in. × $2\frac{1}{3}$ = 22 ft. 11 p. 4 s.
4 „ 6 „	S ≐ 22 ft. 11 p. 4 s. ÷ 6.
9 „ 10 „	

PROBLEM VIII.

Demonstration of the Rule.

It is evident that a rectilineal prismoid is made up of pyramids and wedges having their bases partly in the base and partly in the upper surface of the prismoid, and their height the same as the height of the prismoid; and that a curvilineal prismoid consists of an infinite number of such pyramids and wedges. If we can show, therefore, that the rule holds good in the case of the pyramid and wedge, it will also hold good in the prismoid.

Now, in the case of the pyramid, $4a \doteq A$, and $A' \doteq 0$: therefore $S=\frac{1}{3}A.\,H$ (by Pr. II) $\doteq \frac{1}{6}(A+4a+A')\times H$. In the case of the wedge, L and B being the length and breadth of the base, and L' the length of the edge, the length and breadth of the middle section will be $\frac{1}{2}(L+L')$ and $\frac{1}{2}$ B, and its area, $\frac{1}{4}(L+L')\times B$. Consequently, in this case, $4a \doteq (L+L')\times B$, and $A' \doteq 0$: therefore $S=$

$\frac{1}{6}(2L+L').\ H.B$ (by Pr. VII) $\doteq \frac{1}{6}\{L.B+(L+L').\ B\} \times H = \frac{1}{6}(A+4a+A') \times H.$

Now since the formula of our rule is thus found to apply to every one of the wedges and pyramids composing the prismoid, and since the areas of the ends of the prismoid are the united areas of the ends of the wedges and pyramids, and the area of the middle section of the prismoid is made up of the united areas of the middle sections of the wedges and pyramids, and since H is the same in all, the formula is evidently applicable to the whole prismoid.

Exercise.

$$\begin{aligned} A &\doteq 16 \times 7 &= 112 \\ A' &\doteq 10 \times 5 &= 50 \\ 4a &\doteq 4\,(13 \times 6) &= 312 \\ & & \overline{474} \\ & & 4 \\ \text{Ans. 1 ft. 168 c. in.} & & = \overline{1896} \text{ c. inches.} \end{aligned}$$

Problem XI.

The sphere may evidently be regarded as made up of an infinite number of pyramids having their bases in the surface of the sphere, their vertices in its centre, and their heights all equal to the radius. The solid content of one of these pyramids, by Pr. III, is $\frac{1}{3}A.H = \frac{1}{3}A.\ R$. And, since all the bases added together form the surface of the sphere, $S \doteq \frac{1}{3}$ surface $\times R$. This is *Rule* 2.

Now, by Pr. XVI of Ch. IV, Surface $\doteq D^2 \times \Pi$: therefore $S \doteq \frac{1}{3}D^2.\ \Pi \times R = \frac{1}{6}\Pi \times D^3$. This is *Rule* 1.

Both rules may also be deduced from either of the rules for the following problem, by taking the case in which the segment becomes the half sphere, the height and the radius of the base being then each equal to the radius of the sphere. The content thus found, being doubled, gives Rule 1, from which Rule 2 follows.

A beautiful geometrical demonstration is given in the 21st Pr. of the 3d Book of the Supplement to Playfair's Geometry, along with the previous propositions. It is there shown that every sphere is two-thirds of the circumscribing cylinder. Consequently

$$S \doteq \tfrac{2}{3}D^2 \times \tfrac{1}{4}\Pi \times D = D^3 \times \tfrac{1}{6}\Pi.$$

Cor. The solid content of a hemisphere is two-thirds of

that of a cylinder, and double that of a cone having the same base and height.

Exercise 1.

$$S \doteq (1{\cdot}75)^3 \times {\cdot}5236 = 5{\cdot}359375 \times {\cdot}5236.$$
$$\text{Or, } S \doteq (1\tfrac{3}{4})^3 \times {\cdot}5236 = \tfrac{343}{64} \times {\cdot}5236 = 2{\cdot}806+.$$
$$\text{Then, } 1728 : 2{\cdot}806 :: 1820 : 2{\cdot}955+.$$

Exercise 2.

$$S \doteq 44^3 \times {\cdot}5236 = 85184 \times {\cdot}5236 = 44602+\text{in.}$$

Exercise 3.

$$882\,000^3 \doteq 686{,}128;968{,}000;000{,}000 = C.$$
$$\text{S. of Sun} \doteq C \times {\cdot}5236 = 360{,}480;000{,}000;000{,}000.$$
$$\text{S. of Earth} \doteq 7912^3 \times {\cdot}5236 = 495{,}289;000{,}000 \times {\cdot}5236 = 259{,}333;000{,}000.$$
$$360{,}480;000{,}000 \div 259{,}333 \doteq 1;390{,}000 \text{ nearly.}$$

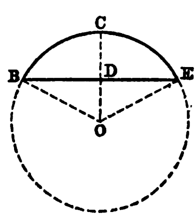
The latter part may be done more simply, thus :—

$$\frac{\text{Sun}}{\text{Earth}} \doteq \frac{882{,}000^3 \times {\cdot}5236}{7912^3 \times {\cdot}5236} = \frac{882\,000^3}{7912^3}.$$

Problem XII.

Demonstration of the Rules.

Put the variable segment (BCE) $= s$, its height (CD) $= v$, and the radius (BD) of its base $= y$. Then



$$\partial s \doteq \Pi\, y^2 \partial v$$
$$= \Pi\,(D - v)\, v\, \partial v$$
$$= \Pi\,(Dv - v^2)\, \partial v.$$
$$\therefore s \doteq \Pi\,(\tfrac{1}{2}D - \tfrac{1}{3}v)\, v^2 \quad (+C=0)$$
$$= \tfrac{1}{6}\Pi\,(3D - 2v)\, v^2 \text{ (which is Rule 2)}$$
$$= \tfrac{1}{6}\Pi\,\{(3D - 3v)\, v + v^2\} \times v$$
$$= \tfrac{1}{6}\Pi\,(3y^2 + v^2) \times v \text{ (which is Rule 1).}$$

Note. Another demonstration might have been given by subtracting the content of the cone BOE from that of

the conical shaped figure BCEO, the content of the latter being proved to be = $\frac{1}{3}$ surf. of seg. × radius of sphere, in the same manner as the Rules in the last problem were demonstrated.

Exercise 1.

By Rule 1,

$$S \doteq (3 \times 16 + 4) \times 2 \times \cdot 5236 = 104 \times \cdot 5236.$$

Exercise 2.

From the Cor. to Pr. LXX of Ch. II,

$$BD^2 \doteq AD \times DC = 2\tfrac{1}{4} \times 4 = 9 = r^2.$$

$$H^2 \doteq (2\tfrac{1}{4})^2 = 5\tfrac{1}{16}.$$

Then, by Rule 1,

$$S \doteq (27 + 5\tfrac{1}{16}) \times 2\tfrac{1}{4} \times \cdot 5236.$$

Exercise 3.

Here we first find OD = 11·9, then DC = 5. Then, by Rule 1,

$$S \doteq (3 \times 144 + 25) \times 5 \times \cdot 5236 = 1196.$$

Problem XIII.

Demonstration of the Rule.

Put KE = H, EF = h, KF = $H+h$, AE = r, and DF = r.

Then (by Pr. XII, Rule 1),

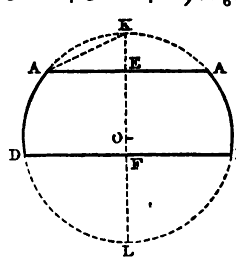
$$\text{Seg. AKA} \doteq (3\mathrm{r}^2 H + H^3) \times \tfrac{1}{6}\Pi, \text{ and}$$

$$\text{Seg. DKD} \doteq \{3r^2 (H+h) + (H+h)^3\} \times \tfrac{1}{6}\Pi.$$

∴ Zone AADD $\doteq$

$$(3r^2H - 3\mathrm{r}^2H + 3r^2h + 3H^2h + 3Hh^2 + h^3) \times \tfrac{1}{6}\Pi.$$

But, DK being supposed drawn,



$$AK^2 \doteq KE \times KL; \text{ and}$$

$$DK^2 \doteq KF \times KL.$$

$$\therefore \frac{AK^2}{KE} \doteq KL = \frac{DK^2}{KF},$$

or $AK^2 \times KF \doteq DK^2 \times KE$.

That is, $(\mathrm{r}^2 + H^2) \times (H+h) \doteq$

$$\{r^2 + (h+H)^2\} \times H.$$

or, $\mathrm{r}^2H + \mathrm{r}^2h + H^3 + H^2h \doteq$

$$r^2H + Hh^2 + 2H^2h + H^3.$$

$$\text{or, } \mathrm{r}^2h \doteq r^2H - \mathrm{r}^2H + H^2h + Hh^2,$$

$$\text{or, } 3\mathrm{r}^2h \doteq 3r^2H - 3\mathrm{r}^2H + 3H^2h + 3Hh^2.$$

If we now substitute $3r^2h$, for its value just found, in the above expression for the zone, we have

$$\text{Zone AADD} = (3\mathrm{r}^2h + 3r^2h + h^3) = \tfrac{1}{6}\Pi.$$

Exercise 1.

$$S \doteq \{3\,(20^2 + 15^2) + 20^2\} \times 20 \times \cdot 5236.$$
$$= 39500 \times \cdot 5236 = 23824 - .$$

Exercise 2.

$$\mathrm{OE} \doteq \sqrt{(65^2 - 52^2)} = 39 \qquad \mathrm{r}^2 \doteq 2704$$
$$\mathrm{OF} \doteq \sqrt{(65^2 - 60^2)} = 25 \qquad r^2 \doteq 3600$$
$$\mathrm{OE} - \mathrm{OF} = h \ldots\ldots \doteq 14 \qquad \mathrm{r}^2 + r^2 \doteq 6304$$

$$(3 \times 6304 + 196) \times 14 \doteq 267512.$$

Problem XVIII.

Let AC, in the following diagram, represent the fixed axis, F, whether transverse or conjugate; and BD, the revolving axis, R.

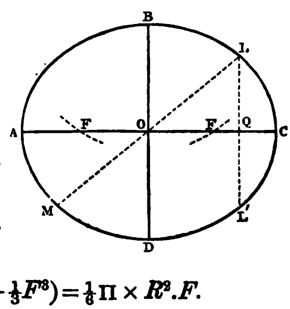
Put s for the variable segment (LCL′) of the spheroid; v for its abscissa (CQ); and y for its ordinate (LQ). Then, by a familiar property of the ellipse,

$$F^2 : R^2 :: \mathrm{AQ} \times \mathrm{QC} = (F - v)\,v : \mathrm{LQ}^2 = y^2.$$

$$\therefore y^2 \doteq \left(\frac{R}{F}\right)^2 \times (Fv - v^2).$$

$$\therefore \partial s = \Pi\, y^2\, \partial v$$
$$\doteq \Pi \left(\frac{R}{F}\right)^2 \times (Fv - v^2)\,\partial v.$$

$$\therefore s \doteq \Pi \left(\frac{R}{F}\right)^2 \times (\tfrac{1}{2}Fv^2 - \tfrac{1}{3}v^3),$$



the constant being $= 0$.

Now let v become F, s becoming S. Then

$$S \doteq \Pi \left(\frac{R}{F}\right)^2 \times (\tfrac{1}{2}F^3 - \tfrac{1}{3}F^3) = \tfrac{1}{6}\Pi \times R^2.F.$$

Cor. 1. Hence it appears that a hemi-spheroid is equal to two-thirds of a cylinder, or to the double of a cone, on the same base and of the same height.

Cor. 2. The prolate spheroid is to the oblate, on the same axes, as the conjugate axis to the transverse.

Exercise 1.

Since the spheroid is oblate, the fixed axis is the shorter. Therefore

$$R \doteq 3\ ;\ F \doteq 2\ ;\ R^2 \times F \times \cdot 5236 \doteq 18 \times \cdot 5236.$$

Exercise 2.

This being a prolate spheroid, the fixed axis is the longer, and therefore

$$R \doteq 2\ ;\ F \doteq 3\ ;\ R^2 \times F \times \cdot 5236 \doteq 12 \times \cdot 5236.$$

Exercise 3.

$$R \doteq 7925 \cdot 5\ ;\ F \doteq 7899\ ;\ R^2 \doteq 62;813,550.$$
$$R^2 \times F \doteq 496,164;23(1,000).$$

PROBLEM XLIII.

No *demonstration* can be given of this which would not involve great intricacy, while it is almost self-evident that no great error can arise if the figure is nearly of a rectangular form, or if it have the peculiarities described in Note 3. But it is almost equally evident that, if it were applied to very irregular figures, the error would be considerable.

Exercise.

Mean dimensions, 49 ft. 6 in., 19 ft. 8 in., 10 ft. 10 in.
Solid content, 10546¼ c. feet.

PROBLEM XLVII.

In Euclid XII, 8, it is shown that similar triangular pyramids are to each other in the triplicate ratio of their homologous sides; that is, (Euc. XI, 33) in the ratio of the cubes of their homologous sides.

Now, of all other solids bounded by plane surfaces, any two that are similar may be divided into the same number of similar pyramids, each pyramid in the one figure having, to the corresponding pyramid in the other figure, the *ratio of the* cubes of any two homologous sides of the

amids, which is the same as that of any two homologous ›s, or other corresponding lines, of the original figures. ısequently the whole solids have to each other the same o (Euc. v, 24).

n the case of solids bounded wholly or partly by curve :aces, the demonstration is the same, the number of ngles being, in this case infinite.

'he truth of our rule, in the cases of the prism, pyramid, ere, and all other figures for whose content we have ticular rules in this chapter, may also be proved from ı particular rules. For, in all of them it may be shown, h more or less trouble in each case, that, of any parlar solid figure, $S \doteq M \times N \times L^3$, L being some lineal ension of the figure, M being some constant multiplier uliar to the class of figures, and N another multiplier uliar to the individual figure and to every similar figure. ısequently, in any similar figure, $s \doteq M \times N \times l^3$. There- $S : s :: M \times N \times L^3 : M \times N \times l^3 :: L^3 : l^3$.

Exercise 1.

$$20^3 : 15^3 :: 6480 : 2734-.$$

Exercise 2.

$$5{\cdot}5^3 : 6^3 :: 15 : 19{\cdot}5-.$$

Exercise 3.

$$1^3 : 8^3 :: 72 \text{ lb.} : 36864 \text{ lb.} = 16 \text{ tons } 9 \text{ c. } 16 \text{ lb.}$$

CHAPTER VI.

PROMISCUOUS EXERCISES.

Exercise 1.

$$3^2 + 4^2 + 12^2 \doteq 169 = 13^2.$$

Exercise 2.

$$(\tfrac{1}{4})^2 : 3^2 :: 1 : 1728.$$

$$\text{Or, } 3^3 . \frac{\Pi}{4} \div \left\{ \left(\frac{1}{4}\right)^3 . \frac{\Pi}{6} \right\} \doteq 3^3 \div \left(\frac{1}{4}\right)^3 = 1728.$$

Exercise 3.

$$\text{Piece wasted} \doteq 10^2 - 10^2 \times \cdot 7854 = 100 - 78\cdot 54.$$

Exercise 4.

The large square would be divided into four squares, each 5 inches in lineal dimensions; and out of each of these a circle would be cut of 5 inches diameter. The part wasted in each small square would be $25 \times (1 - \cdot 7854) = 25 \times \cdot 2146$; and, in all the four pieces, $100 \times \cdot 2146 = 21\cdot 46$.

Exercise 5.

By the reverse formula of Pr. II of Ch. IV,

$$\text{Diagonal} \doteq \sqrt{2A} = \sqrt{36} = 6.$$

Exercise 6.

The surface of the cube consists of six squares, every side of which is the same as one edge of the cube, viz. $\sqrt[3]{125}$, or 5.

$$\therefore \text{Surface} \doteq 6 \times 5^2 = 150.$$

Exercise 7.

Expressing the dimensions in chains, we have

$$\begin{aligned}\text{Area of 1st } \Delta &= \sqrt{(12\cdot 685 \times 7\cdot 685 \times 4\cdot 685 \times \cdot 315)} \\ &= \sqrt{(143\cdot 865 -)} \quad = 11\cdot 994+, \text{ and} \\ \text{Area of 2d } \Delta &= \sqrt{(9 \times 4 \times 4 \times 1)} = 12\cdot 000.\end{aligned}$$

The reason of the slight difference of area caused by so great an error in the measurement of one side, is, that, if we take 800 as the base and construct the two triangles upon that base, the line joining their vertices comes to be very nearly parallel to the base; and, consequently, the two triangles are very nearly equal.

Or, if on a base of 800 we construct an isosceles triangle having each of its equal sides 500,—if we then draw through the vertex a line parallel to the base, and if, with either extremity of the base as a centre and a radius = 500, we describe a circle, we find that it cuts the parallel line in two points, the first of which is, of course, 500 links distant from the other extremity of the base, and the *second is* as nearly as possible 1237.

CHAPTER VII.

UNRESOLVED EXERCISES.

EXERCISES IN CHAPTER III, PROBLEM I.

Answers.

(1), 65.	(4), 2·828+.	(7), 120 yards.
(2), 195.	(5), 1·414+.	(8), 16 feet from it.
(3), 6 ft. 8 in.	(6), 8·385+.	

The solutions of the more difficult are as follows:—

Exercise 6.

We have a ⊾, of which one leg is $7\frac{1}{2}$, and the other $3\frac{3}{4}$. Hence,

$$\text{Hyp.} \doteq \sqrt{(7{\cdot}5^2 + 3{\cdot}75^2)} = \sqrt{(56{\cdot}25 + 14{\cdot}0625)}$$
$$= \sqrt{(70{\cdot}3125)} = 8{\cdot}385 +.$$

Exercise 7.

The footpath divides the field into two ⊾s, in either of which the legs are 216 and 195. Hence we find the hypothenuse 291. The sum of the two legs is $216+195=411$. Therefore the distance saved is $411-291=120$.

Exercise 8.

As the ladder is *first* placed we have a ⊾, of which the hyp. is 65, and the base 33, and, consequently, the perp. 56. Therefore the height of the wall is $56 + 7 = 63$.

In the *proposed* position of the ladder the hyp. is again 65; but the perp. is 63. From these data we find the base 16.

EXERCISES IN CHAPTER III, PROBLEM VI.

Answers.

(1), 81·68+.	(3), 46 ft. 1−in.	(5), 11·727−.
(2), 58·905−.	(4), 0·636+.	(6), 2·997+.

EXERCISES IN CHAPTER III, PROBLEM VII.

Answers.

(1), 6·34375. (2), 42·5. (3), 475·9+.

Solution of Exercise 1.

$$AD \doteq \left(\frac{6\cdot3}{2}\right)^2 \div 2\cdot8 = \frac{6\cdot3 \times 6\cdot3}{4 \times 2\cdot8} = \frac{6\cdot3 \times 0\cdot9}{4 \times 0\cdot4} = 3\cdot54375.$$

To this add CD $= 2\cdot8$.

Exercise 2.

$$\text{Diam.} \doteq \frac{(17 \times 2)^2}{17} + 17 = 68 + 17 = 85.$$

EXERCISES IN CHAPTER III, PROBLEM VIII.

Answers.

(1), 23·4. (3), 44·721 +.
(2), $15\frac{1}{2}$. (4), $1\cdot0\dot{6}$.

In *Exercise* 3, by the second reverse formula,

$$BC \doteq \sqrt{(200 \times 10)} = \sqrt{2000} = 44\cdot721 +.$$

In *Exercise* 4, by the first reverse formula,

$$CD \doteq \frac{2\cdot4 \times 2\cdot4}{5\cdot4} = \frac{\cdot8 \times \cdot4}{\cdot3} = 1\cdot0\dot{6}.$$

EXERCISES IN CHAPTER III, PROBLEM IX.

Answers.—(1), 38; (2), 23·25 −.

EXERCISES IN CHAPTER III, PROBLEM X.

Answers.—(1), 1·658 +; (2), 6·463 +.

EXERCISES IN CHAPTER III, PROBLEM XII.

Answers.

(1), 34 nearly. (4), 63·7 −. (6), 57·4 +.
(2), 3·02 +. (5), 141(8). (7), 34·8 −.
(3), 4·66 −.

In *Exercise* 5, we first find the radius = 1250, and then proceed by Note 2.

In *Exercise* 6, proceeding by Note 1, the supplemental arc is first computed, viz. 5·41, and that is then subtracted from the semicircumf. 62·83 +.

In *Exercise* 7, having found the diameter 18·75, we perceive that the arc is greater than the half circumf., and consequently the half arc greater than a quadrant. Therefore the arc supplemental to the latter must be ascertained by the rule, viz. tabular arc = 1·2857, which, subtracted from 3·1416, gives 1·8559; and that, multiplied by 9·375 gives 17·399 for the half arc. The double of this is the arc required.

Exercises in Chapter III, Problem XIII.

Answers.

(1), 34.	(3), 12·05 +.	(5), 115 −.
(2), 6·8 +.	(4), 1419 −.	

In *Exercise* 2, the chord of the half arc is 3·2515.

In *Exercise* 3, the chord of the whole arc is $2\sqrt{24} = 9{\cdot}800 -$, and the chord of the half arc, $\sqrt{33} = 5{\cdot}745 -$.

In *Exercise* 4, the chord of the half arc is 700.

In *Exercise* 5, the chord dividing the circle into two parts, and the greater arc being evidently a considerable part of the whole circumference, we find the smaller arc, by the rule, and subtract it from the whole circumference. The chord of half the smaller arc is 5·38 +, and that arc itself is 10·79 +. The circumf. of the whole circle is 125·66 +; and the difference of these is 114·87 ±.

Exercise in Chapter III, Problem XIX.

Answers. 14·14 −, and 14·22 +.

Exercises in Chapter III, Problem XXVI.

Answers.

(1), 132·8 in.	(3), 4·243 − in.	(5), the first.
(2), 7·778 +.	(4), $7\frac{1}{8}$ and 5 in.	(6), 6·675 −.

Exercise 3. The extreme length of every scale of chords is the chord of 90°, and the chord of 60° is the radius. The question therefore is simply this:—If the chord of 90° is 1·4142 when the radius is 1, what is the radius when the chord of 90° is 6 inches. We therefore say:—As 1·4142 : 6 :: 1 : 4·243 −.

Exercise 5. The hat of the second ought to be $25\frac{7}{17}$ inches, in order to be proportionate to that of the first.

Exercise 6. The word *area* ought to be *radius*. The proportion will stand thus:—As 68819 : 5·4 :: 85065 : 6·675 –.

Exercises in Chapter IV, Problem I.

Answers.

(1), 645·16 sq. feet.	(6), $213\frac{2}{3}$ sq. feet.
(2), $88\frac{3}{4}$ sq. yards.	(7), £19 : 3 : 3.
(3), 12 sq. inches.	(8), $1\frac{2}{3}$ sq. foot.
(4), 286 sq. feet, and 1664 pages.	(9), 2·898 +.
(5), 81 – 242ds.	(10), 34 yds. $2\frac{1}{3}$ + ft.
	(11), 4·90 – in.

Exercise 7. We have 287 at 9d., making £10 : 15 : 3, and 168 at 1s., making £8 : 8.

Exercise 8. First find the breadth, viz. 10.

Exercise 10. Quarter of an acre contains 1210 sq. yards, the sq. root of which is 34·785 +.

Exercise 11. Each side will be 1 sixth of a square foot = 24 sq. inches, the square root of which is a side of one square, or an edge of the cube.

Exercises in Chapter IV, Problem II.

Answers.—(1), 17·737 –. (2), 9 ft. 4 pts. 8 in.

Exercises in Chapter IV, Problem III.

Answers.—(1), 36. (2), $17\frac{9}{128}$. (3), 9984.

In *Exercise* 3, we first compute the perp. breadth, viz. 96.

Exercises in Chapter IV, Problem IV.

Answers.

(1), 14·5145.	(4), 2574.
(2), $33\frac{21}{40}$.	(5), 3 ro. 8 + p., or 3873 + s. y.
(3), 72 ft. 11 p. 4 in.	(6), 43·45 +.

In Exercise 4, the perpendicular is 117.

In *Exercise* 5, after finding the area = 80028 sq. links and reducing that to roods and poles, we resume the number 80028 sq. links = 8·0028 sq. chains, and multiply the latter number by 484, which is the number of sq. yards in a sq. chain.

Exercise 6. The octagon being divided into eight triangles by lines drawn from the centre to the several angular points, we have the area of one triangle = half the product of its base, 3, by its altitude, which is the radius of the inscribed circle, viz. 3·621. Therefore the area of the octagon will be four times that product, or 12 times 3·621.

Exercises in Chapter IV, Problem V.

Answers.

(1), 504. (3), 14 f. 8 p. 10 – in.
(2), 4 ac. 3 r. 24 + p. (4), 8·714 +.

The product of the half sum and three remainders is, in *Ex.* 1, 254016; in *Ex.* 2, 2403·42 +; in *Ex.* 3, 4501875; in *Ex.* 4, 75·9375.

Exercises in Chapter IV, Problem VI.

Answers.—(1), 1 ac. 2 ro. 6 + pls., or 7447 + sq. yds.
(2), 2·790 – acres.

In *Exercise* 1, the area in square chains is 15·3873, which, multiplied by 484, gives 7447 + sq. yards.

In *Exercise* 2, the products are 267·515 + and 133·217 –. The separate areas are 11·542 – and 16·356 – square chains.

Exercises in Chapter IV, Problem VII.

Answers.

(1), 589·9 –. (2), 14 sq. ft. 105·7 + in. (3), 156·1 + sq. in.

Exercises in Chapter IV, Problem VIII.

Answers.—(1), 10·54 + acres. (2), 14·273 – acres.

Exercise 1.

	acres.		acres.
△ ABC	= 0·8303 −	△ DEF	= 1·1569 +
△ ACD	= 1·9388 −	△ DFH	= 2·2211 +
△ ADH	= 1·9014 +	△ FGH	= 2·4941 −
Field ABCDH	= 4·6705 −	Field DEFGH	= 5·8721 +

Exercise 2.

△ ABC	= 4·5904 − acres.
△ ACD	= 7·6142 −
△ giD	= 0·4575 −
△ Dkl	= 0·6160 +
Trapezoid kmnl	= 1·0217 −
△ Amn	= 0·3029 +
	14·6027 −
Deduct △ Cfg	= 0·3301 −
Total,	14·2726.

Exercises in Chapter IV, Problem XII.

Answers.—(1), ·0490875. (2), 170·87 −

Exercises in Chapter IV, Problem XIII.

Answers.

(1), 170·87 +. (2), 1·735 − ac. (3), 1·039 − ac.

Exercises in Chapter IV, Problem XIV.

Answers.—(1), 20·22 −. (2), 0·7958 − ac.

Exercises in Chapter IV, Problem XV.

Answers.

(1), 11966·13.	(4), 545 +.	(7), 19·1 +.
(2), 2·630 +.	(5), 554(17 −).	(8), 88·6 +.
(3), 0·1451 −.	(6), 553(91 −).	

In *Exercise* 4, the arc, found by Pr. XII of Ch. III, is 34·07 +.

In *Ex.* 5, the arc is $354\cdot\dot{6}$, and the diameter, 625.

In *Ex.* 6, the arc is 354·5.

In *Ex.* 7, the half arc is 2·76966.

In *Ex.* 8, the diameter is 25, and the half arc is 7·09.

Exercises in Chapter IV, Problem XVI.

Answers.

(1), 71·0 +.	(4), 446·4(+).	(7), 1013.
(2), ·168 +.	(5), 920 −.	
(3), 446(−).	(6), 100(9 −).	

In *Exercise* 4, $h \doteq 9\cdot8$.

In *Ex.* 5, proceeding by Rule 6, we first find $\frac{1}{4}c^2 = 903$.

In *Ex.* 6, diameter $\doteq 118\cdot3$.

Exercises in Chapter IV, Problem XVII.

Answers.

(1), 7909 +.	(3), 16·9 +.
(2), 1579 −.	(4), 3046 +.

In *Exercise* 1, radius $\doteq 65$, chord AD $\doteq 64\cdot498$, each segment (by Rule 5) $\doteq 370\cdot5$ +, and trapezoid $\doteq 7168$.

In *Ex.* 2, the radius is 65; the chord AD, 16·125; each segment, 5·49; and the trapezoid 1568.

In *Ex.* 3, OF $\doteq 2\cdot0488$ −, each segment $\doteq 3\cdot34$ −, and trapezoid $\doteq 10\cdot24$ +.

In *Ex.* 4, the breadth is found to be 33·072; each segment, 76·25; and the trapezoid 2893·8.

Exercises in Chapter IV, Problem XVIII.

Answers.—(1), 180·6 +. (2), £348 : 17.

In *Exercise* 2, $R \doteq 135$, $r = 70$, $(R^2 - r^2) \times \Pi = 41862$ sq. ft. $= 4651\frac{1}{3}$ sq. yards.

Exercises in Chapter IV, Problem XIX.

Answers.

(1), 928 +. (2), 2·08 + and 55·52 +, or 22·30 + and 75·74 +.

In *Exercise* 1, finding the areas of the segment by Rule 6, we have

Greater segment = 29·7832.
Smaller segment = 20·5002.

In *Ex.* 2, the areas of the two circles and their segments (found by Rule 6) are as follows:—

Of greater circle.		Of smaller circle.	
Circle........	81·713	Circle,.......	28·274
Smaller seg.	1·944	Smaller seg.	4·027
Greater seg.	79·769	Greater seg.	24·247

Since the two circles may have two different positions, relatively to each other, there arise two values to each result.

Exercises in Chapter IV, Problem XX.

Answers.—(1), 6·061 +. (2), 78·54 −. (3), 1545 +.

In *Exercise* 1, the united area of the ends is ·1754; and that of the sides 5·8860.

In *Ex.* 2, the conv. surface is 39·270; and the united area of the ends, the same.

In *Ex.* 3, the conv. surf. of the roller is 4058·9 inches. By this we divide 6,272,640, being the number of sq. inches in an acre.

Exercises in Chapter IV, Problem XXII.

Answers.

(1), 140·91. (3), 94·25 −.
(2), 282·74 +. (4), 260.

Exercise 2.—Slant height ≐ 13. Conv. surf. = 102·10 +. Area of base = 78·54 −.

Ex. 3.—Area of plate ≐ 200. Circumference of base = 18·8496 +.

Exercises in Chapter IV, Problem XXIII.

Answers.

(1), 432·9. (3), 25·3125.
(2), 5·357 −. (4), 754·0 − sq. in.

Ex. 2.—Area of sides,... 232323 sq. ft.
...... of summit, 1024 ......
Ex. 3.—Circumf. of base, 9·00.
.... of top, 2·25.
Slant height, 4·5.
In *Ex.* 4, the slant height is 10.

Exercises in Chapter IV, Problem XXVI.

Answers.

(1), 63·62 –.
(2), 19·635 –.
(3), 4976 +.
(4), 1·032 –.

Exercises in Chapter IV, Problem XXVII.

Answers.

(1), 97·56 + sq. ft.
(2), 188·5 –.
(3), 706·9 – sq. in.
(4), 207·35 –,
150·79 +, and
94·25 – sq. in.

Exercises in Chapter IV, Problem XXX.

Answers.—(1), 0·4674 + ac. (2), 3·534 +.

Exercises in Chapter IV, Problem L.

Answers.

(1), $1\frac{1}{4}$.
(2), 727·6 –.
(3), 1035 +.
(4), 116 sq. ft. 86 in.

Exercise 3.

$(362 \times 12)^2 : 50^2 :: 1·246 \times 4840 \times 9 \times 144 :$
Or, $181^2 : 25^2 :: 1·246 \times 4840 \times 9$
$: 33922350 \div 32761 = 1035·45.$

Exercise 4.

As $530^2 : 265^2 :: 67160$: area in sq. in.
Or, $4 : 1 :: 67160 : 16790$ sq. in.

Exercises in Chapter V, Problem I.

Answers.

(1), 18 ft. 11 p. 6 s. 8 in.
(2), 65 ft. 2 p. 10 in.
(3), 300 tons.
(4), 1578 –.
(5), 15·42 –.
(6), 4·38 +.
(7), 2·38 +.

Exercises in Chapter V, Problem II.

Answers.

(1), ·03316 – ft. (2), ·0328 +. (3), 8·482 +.
In *Exercise* 2. Area of base $\doteq$ 29·2195 sq. in.
In *Ex.* 3, the radius of the base is 1·5.

Exercises in Chapter V, Problem III.

Answers.—(1), 24·68 +. (2), 1794 +.

Exercise 1.

$$\begin{array}{rl} \text{Area of base,} & 9 \times a \\ \text{...... of summit,} & 4 \times a \\ \sqrt{(9a \times 4a)} \doteq & \underline{6 \times a} \\ & 19 \times a = 49{\cdot}364 -. \end{array}$$

Exercise 2.

We first find, difference of diameters = 11·2, and ∴ diam. of greater end = 21·2. Then putting a for ·7854,

$$\begin{array}{lr} \text{Area of greater end} \doteq 21{\cdot}2^2 \times a = & 449{\cdot}44a \\ \text{Area of smaller end} \doteq \quad 10^2 \times a = & 100{\cdot}00a \\ \sqrt{(21{\cdot}2^2 a \times 10^2 a)} \quad \doteq \ldots\ldots\ldots\ldots & \underline{212{\cdot}00a} \\ & 761{\cdot}44a \end{array}$$

∴ Content of frust. $\doteq 761{\cdot}44a \times 3 = 2284{\cdot}32a$.

Exercise in Chapter V, Problem VII.

Answer.—64208 +.

Exercise in Chapter V, Problem VIII.

Answer.—·754 – of a yard, and ·942 + of a ton.

Exercises in Chapter V, Problem XI.

Answers.

(1), 5277 – millions of cubic miles.
(2), $29\frac{1}{3}$ + lb. (3), 728·1 + lb.

Exercises in Chapter V, Problem XII.

Answers.

(1), 2136 +. (2), 25205. (3), 2·736 –.

Exercises in Chapter V, Problem XIII.

Answers.—(1), 14·89 +. (2), 384·0 + c. feet.

Exercises in Chapter V, Problem XVIII.

Answers.—(1), 18·85 –. (2), 25·13 +.

Exercise in Chapter V, Problem XLIII.

Answer.—£2 : 14 : $6\frac{3}{4}$ –. The solid content is 803·45 c. feet, or 29·7574 c. yards.

Exercises in Chapter V, Problem XLVII.

Answers.

(1), 3 lb. 0·13 – oz.
(2), 30·20 + cwt.
(3), 49·16 – times.
(4), 60 st. 13 – lb., and 36 st. 8 – lb.
(5), Upper part, ·13 – lb. Lower part, 3·37 + lb.

Exercise 3.

As $2160^3 : 7912{\cdot}5^3 :: 1$, or
As $17280^3 : 63300^3 :: 1$, or
As $864^3 : 3165^3 :: 1 : 49{\cdot}16 -$.

Exercise 4.

As $(6\frac{1}{2})^3 : (5\frac{3}{4})^3 ::$ 88 st., or
As $26^3 : 23^3 ::$ 88 st. : 60 st. 13 – lb., and
As $26^3 : 23^3 ::$ 739 lb. : 512 – lb.

Exercise 5.

As $15^3 : 5^3 ::$ 3·5 lb., or
As $3^3 : 1^3 ::$ 3·5 lb. : weight of upper part.

Exercises in Chapter VI.

Answers.

(1), 13·84.
(2), 2 ft. 11 in.
(3), 36·50 – ft.
(4), Almost exactly 30 tons.
(5), Breadth, 4 ft. 5·0 – in.
Height, 8·6 – inches.
Floor, 91·8 + sq. ft.
Glass, 105 + sq. ft.
Content, 41·8 + c. ft.

Exercise 1.

The diagonals of a rhombus bisect each other at right-angles. The rhombus is therefore divided into four $\triangle$s, in any one of which we have given the diagonal 8·65, and one leg 5·19, from which we find the other leg 6·92. The double of this is the longer diagonal.

Exercise 2.

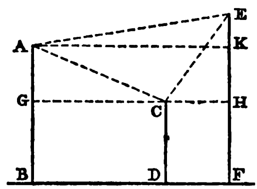
In every case of reflection by a plane mirror the reflected figure appears of the same size as the real figure, and as much behind the mirror as the real object is before it. The mirror, therefore, being placed vertically is half way between the eye of the spectator and his reflected figure; and, consequently, if st. lines be supposed to extend from his eye to the highest and lowest points of the reflected

figure, they will intercept a length upon the mirror equal to half the height of the reflected figure, which is the same as that of the real figure. The looking-glass must therefore be half the height of the person using it, that he may see himself from head to foot.

Exercise 3.

AB, CD, and EF being the three poles and the other lines being drawn as in the diagram, we have, in $\triangle$ EHC,

$CE = 20$, $CH = 12$,
and $\therefore EH = 16$.
$\therefore HF \doteq 15 = BG$,
and $AG \doteq 25 - 15 = 10$.
Hence, in $\triangle$ AGC,
$AC \doteq 26$, $AG \doteq 10$,
and $\therefore CG \doteq 24$.



Lastly, in $\triangle$ AKE,

$$AK \doteq GH = GC + CH = 24 + 12 = 36,$$
$$\text{and } KE \doteq EF - AB = 31 - 25 = 6.$$
$$\therefore AE \doteq \sqrt{1332} = 36{\cdot}50 - .$$

Exercise 4.

The area cleared was $= 10$ yds. $\times 28$ in. $= 10 \times 3 \times 12 \times 28$ in. Then

As $10 \times 3 \times 12 \times 28 : 4840 \times 9 \times 144 : 108$, or
As $7 : 484 \times 3 \times 3 :: 108$: weight in pounds.

$$\therefore \text{Weight in tons} \doteq \frac{484 \times 3 \times 3 \times 108}{7 \times 112 \times 20} = \frac{121 \times 3 \times 3 \times 27}{7 \times 28 \times 5}$$

$= 30$, almost exactly.

CHAPTER VIII.

PRODUCTS IN FREQUENT USE.

NOTE. This Chapter, like the last Chapter in Part I, has no counterpart in the "Course," but may be useful to the teacher for checking calculations relating to circles, spheres, &c. The products are not those of Π, $\frac{1}{4}\Pi$, &c., taken strictly true, even as far as they go, but of the abbreviated numbers $3{\cdot}1416$, ${\cdot}7854$, &c. They never

differ, however, from the true products except occasionally in the last figure, such difference, when it does occur, being indicated by inclosing that figure in a parenthesis.

Products of 3·1416.

$3{\cdot}1416 \times 1 \doteq$	3·1416
$2 =$	6·2832
$3 =$	9·4248
$4 =$	12·5664
$5 =$	15·7080
$6 =$	18·8496
$7 =$	21·991(2)
$8 =$	25·132(8)
$9 =$	28·274(4).

$3{\cdot}1416 \times \frac{1}{2} \doteq$	1·57080
$\frac{1}{3} =$	1·04720
$\frac{1}{4} =$	0·78540
$\frac{1}{5} =$	0·62832
$\frac{1}{6} =$	0·52360
$\frac{1}{7} =$	0·44880
$\frac{1}{8} =$	0·39270
$\frac{1}{9} =$	0·34907 –.

Products of ·7854.

$\cdot 7854 \times 1 \doteq$	0·7854
$2 =$	1·5708
$3 =$	2·3562
$4 =$	3·1416
$5 =$	3·9270
$6 =$	4·7124
$7 =$	5·4978
$8 =$	6·2832
$9 =$	7·0686

$\cdot 7854 \times \frac{1}{2} \doteq$	·39270
$\frac{1}{3} =$	·26180
$\frac{1}{4} =$	·19635
$\frac{1}{5} =$	·15708
$\frac{1}{6} =$	·13090
$\frac{1}{7} =$	·11220
$\frac{1}{8} =$	·09817 +
$\frac{1}{9} =$	·08727 –.

Products of ·07958.

$\cdot 07958 \times 1 \doteq$	·07958
$2 =$	·1591(6)
$3 =$	·2387(4)
$4 =$	·3183(2)
$5 =$	·3979(0)
$6 =$	·4774(8)
$7 =$	·5570(6)
$8 =$	·6366(4)
$9 =$	·7162(2).

$\cdot 07958 \times \frac{1}{2} \doteq$	·03979
$\frac{1}{3} =$	·02658 –
$\frac{1}{4} =$	·01989(5)
$\frac{1}{5} =$	·01591(6)
$\frac{1}{6} =$	·01326 +
$\frac{1}{7} =$	·01137 –
$\frac{1}{8} =$	·00995 –
$\frac{1}{9} =$	·00884 +.

Products of ·5236.

$\cdot 5236 \times 1 \doteq$	0·5236
$2 =$	1·0472
$3 =$	1·5708
$4 =$	2·0944
$5 =$	2·6180
$6 =$	3·1416
$7 =$	3·6652
$8 =$	4·1888
$9 =$	4·7124

$\cdot 5236 \times \frac{1}{2} \doteq$	·26180
$\frac{1}{3} =$	·17453 +
$\frac{1}{4} =$	·13090
$\frac{1}{5} =$	·10472
$\frac{1}{6} =$	·08727 –
$\frac{1}{7} =$	·07480
$\frac{1}{8} =$	·06545
$\frac{1}{9} =$	·05818 –.

END OF PART SECOND.

MURRAY AND GIBB, PRINTERS, EDINBURGH.